Limits and continuity

Submission instructions: Clearly write your full name and ID number on the first page. To avoid marking and handling difficulties, please staple all submitted pages together and answer the questions in the order they appear on the assignment.

Academic integrity: Students are encouraged to collaborate on assignment problems but must write up their assignments independently. Copying is strictly forbidden!

Suggested problems:

Section 2.1, 1 – 12
Section 2.2, 1 – 34 (odd)
Section 2.3, 1 – 34 (odd)
Section 2.5, 1 – 28 (odd)

Problems for submission:

1. Evaluate the following limits

   (a) \( \lim_{x \to 1} \frac{x^2 - 2x + 1}{x^2 - 6x + 8}, \)

   (b) \( \lim_{x \to 2} \frac{x^2 - 5x + 6}{x - 2}, \)

   (c) \( \lim_{x \to \infty} \frac{x^2 + 2x + 1}{-2x^2 + x}, \)

   (d) \( \lim_{x \to \infty} \frac{3x - 2x}{3x + 2x}, \)

   (e) \( \lim_{x \to \infty} 2x - \sqrt{4x^2 + x - 1}, \)

   (f) \( \lim_{h \to 0} \frac{\sqrt{x + h} - \sqrt{x}}{h}. \)
2. Use the identity \( \lim_{x \to 0} \frac{\sin(x)}{x} = 1 \) to evaluate the following limits:

(a) \( \lim_{x \to 0} \frac{1 - \cos^2(x)}{x^2} \),

(b) \( \lim_{t \to \infty} t \sin \left( \frac{1}{2t} \right) \).

3. Use the squeeze theorem to evaluate the following limits:

(a) \( \lim_{x \to \infty} e^{-x} \sin(x) \),

(b) \( \lim_{x \to \infty} \frac{1 - \cos(x)}{\ln(x)} \).

4. Determine values of \( a \) and \( b \) which make the following function continuous on \( x \in \mathbb{R} \):

\[
f(x) = \begin{cases} 
ax + 1, & \text{for } x < -1 \\
2x^2 + 1 - 2, & \text{for } -1 \leq x < b \\
2, & \text{for } b \leq x.
\end{cases}
\]

5. Use the Intermediate Value Theorem to show that \( f(x) = x^3 - 3x^2 + 3x - \sqrt{x} \) has roots in the intervals \( 0 < x < 1 \) and \( 1 < x < 2 \). (Hint: It is not enough to find points \( x = a \) such that \( f(a) = 0 \)!)

6. Not only does the Intermediate Value Theorem allow us to guarantee the existence of a root for continuous functions, it allows us to bound the root. If \( f(a) < 0 \) and \( f(b) > 0 \) (or vice versa), then the root \( x^* \) satisfies \( a < x^* < b \) so that, if the distance between \( a \) and \( b \) is small, this interval can be taken as a reasonable estimate of the root. The question then becomes whether we can use the IVT to get successively better approximations (i.e. “smaller” and “smaller” intervals) of the actual value of the root.

The answer, of course, is that we can. Imagine an interval where we know a root exists by the IVT (say \( a < x^* < b \)). Now pick the midpoint of the interval and evaluate \( f \left( \frac{a + b}{2} \right) \). This value is either equal to zero (in which case, we have found the root!) or has an opposite sign of one of the two original end points (in which case, the root must lie in that half of the original interval!). We can apply this successively to each new interval we find and consequently get arbitrarily close to a root in a finite number of iterations.

The method is called the Bisection Method. Use it to estimate the root of \( f(x) = e^{-x} - x \) to two decimal places of accuracy.