1. Evaluate the following limits

(a) \[ \lim_{{x \to 1}} \frac{x^2 - 2x + 1}{x^2 - 6x + 8}, \]

**Solution:** We plug in the value \( x = 1 \) to get
\[ \lim_{{x \to 1}} \frac{x^2 - 2x + 1}{x^2 - 6x + 8} = \frac{(1)^2 - 2(1) + 1}{(1)^2 - 6(1) + 8} = \frac{0}{3} = 0. \]

(b) \[ \lim_{{x \to 2}} \frac{x - 2}{x^2 - 5x + 6}, \]

**Solution:** If we attempt to evaluate at \( x = 2 \) we get "0/0" which is an indeterminate form. Instead, we factor to get
\[ \lim_{{x \to 2}} \frac{x - 2}{x^2 - 5x + 6} = \lim_{{x \to 2}} \frac{x - 2}{(x - 2)(x - 3)} = \lim_{{x \to 2}} \frac{1}{x - 3} = -1. \]

(c) \[ \lim_{{x \to \infty}} \frac{x^2 + 2x + 1}{-2x^2 + x}. \]

**Solution:** We have
\[ \lim_{{x \to \infty}} \frac{x^2 + 2x + 1}{-2x^2 + x} = \frac{x^2 + 2x + 1}{-2x^2 + x} \left( \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \right) \]
\[ = \frac{1 + \frac{2}{x} + \frac{1}{x^2}}{-2 + \frac{1}{x}} \]
\[ = \frac{-2 + \frac{1}{x}}{1} \]
\[ = -1. \]

(d) \[ \lim_{{x \to \infty}} \frac{3x - 2^x}{3x + 2^x}, \]

**Solution:** The behaviour cannot be evaluated by simply taking \( x \) to go to infinity in each term. Instead, we divide the numerator
and denominator by the largest term in $x$ in order to drive the troublesome limits to zero. This gives

$$\lim_{x \to \infty} \frac{3^x - 2^x}{3^x + 2^x} = \lim_{x \to \infty} \frac{3^x - 2^x}{3^x + 2^x} \left( \frac{1}{3^x} \right) \left( \frac{\frac{1}{3^x}}{\frac{1}{3^x}} \right) = \lim_{x \to \infty} \frac{1 - \left( \frac{2}{3} \right)^x}{1 + \left( \frac{2}{3} \right)^x} = 1.$$

(e) \( \lim_{x \to \infty} 2x - \sqrt{4x^2 + x - 1} \),

**Solution:** We have a “$\infty - \infty$” limit, which is an indeterminate form. We rationalize the square roots to get

$$\lim_{x \to \infty} 2x - \sqrt{4x^2 + x - 1} = \lim_{x \to \infty} \frac{2x - \sqrt{4x^2 + x - 1}}{2} \left( \frac{2x + \sqrt{4x^2 + x - 1}}{2x + \sqrt{4x^2 + x - 1}} \right)$$

$$= \lim_{x \to \infty} \frac{4x^2 - (4x^2 + x - 1)}{2x + \sqrt{4x^2 + x - 1}}$$

$$= \lim_{x \to \infty} \frac{-x + 1}{2x + \sqrt{4x^2 + x - 1}} \left( \frac{1}{x} \right)$$

$$= \lim_{x \to \infty} \frac{-1 + \frac{1}{x}}{2 + \sqrt{4 + \frac{1}{x} - \frac{1}{x^2}}}$$

$$= \left( \frac{-1}{4} \right).$$

(f) \( \lim_{h \to 0} \frac{\sqrt{x + h} - \sqrt{x}}{h} \).
Solution: We have

\[
\lim_{h \to 0} \frac{\sqrt{x + h} - \sqrt{x}}{h} = \lim_{h \to 0} \frac{\sqrt{x + h} - \sqrt{x}}{h} \frac{(\sqrt{x + h} + \sqrt{x})}{(\sqrt{x + h} + \sqrt{x})}
\]

\[
= \lim_{h \to 0} \frac{x + h - x}{h(\sqrt{x + h} + \sqrt{x})}
\]

\[
= \lim_{h \to 0} \frac{h}{h(\sqrt{x + h} + \sqrt{x})}
\]

\[
= \lim_{h \to 0} \frac{1}{\sqrt{x + h} + \sqrt{x}}
\]

\[
= \frac{1}{2\sqrt{x}}.
\]

2. Use the identity \( \lim_{x \to 0} \frac{\sin(x)}{x} = 1 \) to evaluate the following limits:

(a) \( \lim_{x \to 0} \frac{1 - \cos^2(x)}{x^2} \),

(b) \( \lim_{t \to \infty} t \sin \left( \frac{1}{2t} \right) \).

Solution (a): We have

\[
\lim_{x \to 0} \frac{1 - \cos^2(x)}{x^2} = \lim_{x \to 0} \frac{\sin^2(x)}{x^2} = \left( \lim_{x \to 0} \frac{\sin(x)}{x} \right) \left( \lim_{x \to 0} \frac{\sin(x)}{x} \right) = (1)(1) = 1.
\]

Solution (b): We use the substitution \( x = 1/2t \) and notice that \( x \to 0 \) as \( t \to \infty \). This gives

\[
\lim_{t \to \infty} t \sin \left( \frac{1}{2t} \right) = \lim_{x \to 0} \frac{\sin(x)}{2x} = \frac{1}{2} \lim_{x \to 0} \frac{\sin(x)}{x} = \frac{1}{2}.
\]

3. Use the squeeze theorem to evaluate the following limits:

(a) \( \lim_{x \to \infty} e^{-x} \sin(x) \).

Solution: We have

\[
-1 \leq \sin(x) \leq 1 \implies -e^{-x} \leq e^{-x} \sin(x) \leq e^{-x}
\]

\[
\implies -\lim_{x \to \infty} e^{-x} \leq \lim_{x \to \infty} e^{-x} \sin(x) \leq \lim_{x \to \infty} e^{-x} \sin(x)
\]
\[ 0 \leq \lim_{x \to \infty} e^{-x} \sin(x) \leq 0. \]

It follows that
\[ \lim_{x \to \infty} e^{-x} \sin(x) = 0. \]

(b) \[ \lim_{x \to \infty} \frac{1 - \cos(x)}{\ln(x)}. \]

Solution: We have
\[ \lim_{x \to \infty} \ln(x) = \infty \implies \lim_{x \to \infty} \frac{1}{\ln(x)} = 0. \]

We also have
\[ -1 \leq \cos(x) \leq 1 \implies 0 \leq 1 - \cos(x) \leq 2. \]
\[ \implies 0 \leq \frac{1 - \cos(x)}{\ln(x)} \leq \frac{2}{\ln(x)}. \]
\[ \implies 0 \leq \lim_{x \to \infty} \frac{1 - \cos(x)}{\ln(x)} \leq \lim_{x \to \infty} \frac{2}{\ln(x)} = 0. \]

It follows that
\[ \lim_{x \to \infty} \frac{1 - \cos(x)}{\ln(x)} = 0. \]

4. Determine values of \( a \) and \( b \) which make the following function continuous on \( x \in \mathbb{R} \):

\[ f(x) = \begin{cases} 
ax + 1, & \text{for } x < -1 \\
2^{x+1} - 2, & \text{for } -1 \leq x < b \\
2, & \text{for } b \leq x.
\end{cases} \]

Solution: To make this function continuous at \( x = -1 \), we need to have
\[ \lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x) \implies a(-1) + 1 = 2^0 - 2 \]
\[ \implies -a + 1 = -1 \implies a = 2. \]

To make this function continuous at \( x = b \) we need to have
\[ \lim_{x \to b^-} f(x) = \lim_{x \to b^+} f(x) \implies 2^{b+1} - 2 = 2 \]
\[ 2^{b+1} = 4 \implies b + 1 = \log_2(4) = 2 \implies b = 1. \]
5. Use the Intermediate Value Theorem to show that \( f(x) = x^3 - 3x^2 + 3x - \sqrt{x} \) has roots in the intervals \( 0 < x < 1 \) and \( 1 < x < 2 \). (Hint: It is not enough to find points \( x = a \) such that \( f(a) = 0 \! \) )

**Solution:** We check the value of \( f(x) \) at the points \( x = 0, x = 1 \) and \( x = 2 \) to get

\[
\begin{align*}
  f(0) &= 0 \\
  f(1) &= 0 \\
  f(2) &\approx 0.586.
\end{align*}
\]

We notice that we cannot use the Intermediate Value Theorem to guarantee a root in either of the intervals \( 0 < x < 1 \) or \( 1 < x < 2 \) because the sign of \( f(x) \) does not change at the end points (even though the end points are roots themselves!). We need to choose different endpoints. We instead choose the interval \( 0.1 < x < 0.9 \) and \( 1.1 < x < 2 \) to get

\[
\begin{align*}
  f(0.1) &\approx -0.045 \\
  f(0.9) &\approx 0.050 \\
  f(1.1) &\approx -0.048 \\
  f(2) &\approx 0.586.
\end{align*}
\]

We notice that the sign does change at the endpoints of the intervals \( 0.1 < x < 0.9 \) and \( 1.1 < x < 2 \), so that there is a root guaranteed in the intervals by the IVT. Since these intervals are strictly contained with the intervals \( 0 < x < 1 \) and \( 1 < x < 2 \), the result follows.

6. Not only does the Intermediate Value Theorem allow us to guarantee the existence of a root for continuous functions, it allows us to bound the root. If \( f(a) < 0 \) and \( f(b) > 0 \) (or vice versa), then the root \( x^* \) satisfies \( a < x^* < b \) so that, if the distance between \( a \) and \( b \) is small, this interval can be taken as a reasonable estimate of the root. The question then becomes whether we can use the IVT to get successively better approximations (i.e. “smaller” and “smaller” intervals) of the actual value of the root.

The answer, of course, is that we can. Imagine an interval where we know a root exists by the IVT (say \( a < x^* < b \)). Now pick the midpoint of the interval and evaluate \( f \left( \frac{a + b}{2} \right) \). This value is either equal to zero (in which case, we have found the root!) or has an opposite sign
of one of the two original end points (in which case, the root must lie in that half of the original interval!). We can apply this successively to each new interval we find and consequently get arbitrarily close to a root in a finite number of iterations.

The method is called the *Bisection Method*. Use it to estimate the root of \( f(x) = e^{-x} - x \) to two decimal places of accuracy.

**Solution:** We know from class that there is a root of \( f(x) \) in the interval \( 0 < x < 1 \). In particular, we have

\[
\begin{align*}
  f(0) &= 1 \\
  f(1) &\approx -0.632.
\end{align*}
\]

The bisection method tells us to consider the midpoint of the interval, so we check \( x = 0.5 \). This gives us

\[
\begin{align*}
  f(0.5) &= 0.102
\end{align*}
\]

so that our new interval is \( 0.5 < x < 1 \). The midpoint of this interval is \( 0.75 \) so we check

\[
\begin{align*}
  f(0.75) &= -0.278
\end{align*}
\]

so that our new interval is \( 0.5 < x < 0.75 \). The midpoint of the interval is \( 0.625 \) so we check

\[
\begin{align*}
  f(0.625) &= -0.0897
\end{align*}
\]

so that our new interval is \( 0.5 < x < 0.625 \). The midpoint of the interval is \( 0.5625 \) so we check

\[
\begin{align*}
  f(0.5625) &= 0.00728
\end{align*}
\]

so that our new interval is \( 0.5625 < x < 0.625 \). The midpoint of the interval is \( 0.59375 \) so we check

\[
\begin{align*}
  f(0.59375) &= -0.0415
\end{align*}
\]

so that our new interval is \( 0.5625 < x < 0.59375 \). The midpoint of the interval is \( 0.578125 \) so we check

\[
\begin{align*}
  f(0.578125) &= -0.0172
\end{align*}
\]
so that our new interval is $0.5625 < x < 0.578125$. The midpoint of the interval is 0.5703125 so we check

$$f(0.5703125) = -0.00496$$

so that our new interval is $0.5625 < x < 0.5703125$. The midpoint of the interval is 0.56640625 so we check

$$f(0.56640625) = 0.00116$$

so that our new interval is $0.56640625 < x < 0.5703125$. Since every value in this interval rounds to 0.57, we can finally say we have found the answer to two decimal places of accuracy! (Whew! That was a lot of work, but we will find a better way to do this process in a few weeks. The actual value, to the precision of my calculator, is $x = 0.5671432904$.)