MATH 116, Spring 2012, Assignment 6
Solutions
Product, quotient, and chain rules; implicit and logarithmic differentiation

1. Find $f'(x)$ for the following functions:

(a) $f(x) = x^2 \arcsin(x)$

Solution: By the product rule, we have

$$\frac{d}{dx} [x^2 \arcsin(x)] = 2x \arcsin(x) + \frac{x^2}{\sqrt{1-x^2}}.$$

(b) $f(x) = \frac{\tan^2(x)}{2}$

Solution: By the chain rule, we have

$$\frac{d}{dx} \left[ \frac{\tan^2(x)}{2} \right] = \tan(x) \sec^2(x).$$

(c) $f(x) = \ln\left( \frac{\sqrt{x^2+1}}{x} \right)$

Solution: By the chain rule and quotient rule, we have

$$\frac{d}{dx} \left[ \ln\left( \frac{\sqrt{x^2+1}}{x} \right) \right] = \left( \frac{x}{\sqrt{x^2+1}} \right) \left( \frac{x^2}{x^2+1} - \frac{\sqrt{x^2+1}}{x} \right)$$

$$= \frac{x}{\sqrt{x^2+1}} \left( \frac{x^2 - x^2 - 1}{x^2 \sqrt{x^2+1}} \right)$$

$$= -\frac{1}{x(x^2+1)}.$$

(d) $f(x) = x \cos(e^x)$

Solution: By the product rule and chain rule, we have

$$\frac{d}{dx} [x \cos(e^x)] = \cos(e^x) - x \sin(e^x) e^x.$$
2. Find \( \frac{dy}{dx} \) for the following relations:

(a) \( xy^2 + x^2y = 1 \)

Solution: Using implicit differentiation, we have

\[
xy^2 + x^2y = 1 \\
\implies y^2 + 2xyy' + 2xy + x^2y' = 0 \\
\implies y'(2xy + x^2) = -y^2 - 2xy \\
\implies y' = \frac{-y(y + 2x)}{x(2y + x)}.
\]

(b) \( \sin(y) = \sqrt{xy} \)

Solution: Using implicit differentiation, we have

\[
\sin(y) = \sqrt{xy} \\
\implies \cos(y)y' = \frac{y + xy'}{2\sqrt{xy}} \\
\implies \sqrt{xy}\cos(y)y' - xy' = y \\
\implies y' = \frac{y}{\sqrt{xy}\cos(y) - x}.
\]

(c) \( \frac{x + y}{xy} = x \)

Solution: Using implicit differentiation, we have

\[
\frac{x + y}{xy} = x \\
\implies \frac{(1 + y')(xy) - (x + y)(y + xy')}{(xy)^2} = 1 \\
\implies xy + xyy' - xy - x^2y' - y^2 - xyy' = x^2y^2 \\
\implies y' = -\frac{y^2(1 + x^2)}{x^2}.
\]

3. Suppose we know \( p(0) = 1, \ p'(0) = 2, \) and \( q'(1) = 3. \) Determine \( r'(0) \) for

\[ r(t) = q(p(t)). \]
Solution: By the chain rule, we have
\[ r'(t) = q'(p(t))p'(t). \]

It follows from the given information that
\[ r'(0) = q'(p(0))p'(0) = 2q'(1) = 6. \]

4. Determine the equation of the tangent line to \((x, y) = (5/4, 3/4)\) for the relation
\[ x^2 - y^2 = 1. \]

Solution: By implicit differentiation, we have
\[ x^2 - y^2 = 1 \]
\[ \implies 2x - 2yy' = 0 \]
\[ \implies y' = \frac{x}{y}. \]

We need to find the equation to the tangent line \(y = mx + b\). At the point \((x, y) = (5/4, 3/4)\) we have
\[ y'(5/4, 3/4) = \frac{5}{4} \cdot \frac{5}{4} = \frac{5}{3} \]
so that \(m = 5/3\). Evaluating at the point \((5/4, 3/4)\) gives
\[ \frac{3}{4} = \frac{5}{3} \cdot \frac{5}{4} + b \implies b = \frac{3}{4} - \frac{25}{12} = -\frac{4}{3}. \]

It follows that we have
\[ y = \frac{5}{3}x - \frac{4}{3}. \]

5. Determine \(y'\) for the following relations:

(a) \(y = \ln(x)^x\)

Solution: By logarithmic differentiation, we have
\[ y = \ln(x)^x \]
\[ \implies \ln(y) = x \ln(\ln(x)) \]
\[ \implies \frac{y'}{y} = \ln(\ln(x)) + \frac{x}{\ln(x)x} \]
\[ \implies y' = y \left( \ln(\ln(x)) + \frac{1}{\ln(x)} \right) \]
\[ \implies y' = \ln(x)^x \left( \ln(\ln(x)) + \frac{1}{\ln(x)} \right). \]
(b) \( y = x^{x^x} \)

**Solution:** By logarithmic differentiation, we have

\[
y = x^{x^x} \\
\implies \ln(y) = \ln(x^{x^x}) = x^x \ln(x) \\
\implies \frac{y'}{y} = \left[ \frac{d}{dx} x^x \right] \ln(x) + x^x \left[ \frac{d}{dx} \ln(x) \right].
\]

We know what the derivative of \( x^x \) is from lecture. We have

\[
\left[ \frac{d}{dx} x^x \right] = x^x (\ln(x) + 1).
\]

It follows that

\[
\frac{y'}{y} = x^x (\ln(x) + 1) \ln(x) + \frac{x^x}{x} \\
\implies y' = x^{x^x} \left[ x^x (\ln(x) + 1) \ln(x) + x^{x-1} \right].
\]

(c) \( y = \frac{e^{x(x^2 - 1)^3}}{\sqrt{x^3 + x + 1}} \)

**Solution:** By logarithmic differentiation, we have

\[
y = \frac{e^{x(x^2 - 1)^3}}{\sqrt{x^3 + x + 1}} \\
\implies \ln(y) = \ln \left( \frac{e^{x(x^2 - 1)^3}}{(x^3 + x + 1)^{1/2}} \right) \\
\implies \ln(y) = x + 3 \ln(x^2 - 1) - \frac{1}{2} \ln(x^3 + x + 1) \\
\implies \frac{y'}{y} = 1 + \frac{6x}{x^2 - 1} - \frac{3x^2 + 1}{2(x^3 + x + 1)} \\
\implies y' = \frac{e^{x(x^2 - 1)^3}}{(x^3 + x + 1)^{1/2}} \left( 1 + \frac{6x}{x^2 - 1} - \frac{3x^2 + 1}{2(x^3 + x + 1)} \right) \\
\implies y' = \frac{e^{x(x^2 - 1)^3}}{(x^3 + x + 1)^{3/2}} (2x^5 + 9x^4 + 16x^2 + 10x - 1)
\]

(You do not need to fully simplify!)

6. Consider the function \( f(x) = \cos(x) - 2x \).
(a) Prove that \( f(x) \) has at least one root.

**Solution:** The function is continuous so we consider using the intermediate value theorem. Evaluating at \( x = 0 \) and \( x = \pi/4 \), we have

\[
f(0) = 1 > 0.
\]

and

\[
f(\pi/4) = -\frac{\pi}{2} < 0.
\]

It follows the the IVT that there is at least one root in the interval \((0,1)\).

(b) Prove that \( f(x) \) has at most one root.

We consider the derivative of \( f(x) \). We have

\[
f'(x) = -\sin(x) - 2
\]

and notice that, since \(-1 \leq \sin(x) \leq 1\), we have that \(-3 \leq f'(x) \leq -1\). Most importantly, the derivative is always negative. This means once the function crosses the \( x \)-axis, it cannot reverse course and cross it again, since this would require a positive derivative for at least some portion of the domain. It follows that there is at most one root. Combining part (a) and (b), we can conclude that there is exactly one root.