1. Use your knowledge of derivatives to evaluate the following indefinite integrals (remember to add your arbitrary constants!):

(a) \( \int x^{2012} \, dx \),

**Solution:** We have

\[ \int x^{2012} \, dx = \frac{x^{2013}}{2013} + C. \]

(b) \( \int (x + 1)^2 \, dx \),

**Solution:** We have

\[ \int (x + 1)^2 \, dx = \int (x^2 + 2x + 1) \, dx = \frac{x^3}{3} + x^2 + x + C. \]

(c) \( \int \frac{3x^2}{x^3 + 1} \, dx \),

**Solution:** For this case we will have to guess. We notice that the derivative of the denominator appears in the numerator. We might suspected that this has occurred as a result of the chain rule. The function which puts its argument in the denominator when we take the derivative is \( \ln(x) \), so we try \( \ln(x^3 + 1) \). We have

\[ \frac{d}{dx} \ln(x^3 + 1) = \frac{1}{x^3 + 1} \cdot (3x^2) = \frac{3x^2}{x^3 + 1}. \]

It follows that

\[ \int \frac{3x^2}{x^3 + 1} \, dx = \ln(x^3 + 1) + C. \]
(d) \( \int \frac{1}{4 + x^2} \, dx \),

**Solution:** We have

\[
\int \frac{1}{4 + x^2} \, dx = \frac{1}{4} \int \frac{1}{1 + \left(\frac{x}{2}\right)^2} \, dx.
\]

We guess the form \( \arctan(x/2) \) to get

\[
\frac{d}{dx} \arctan \left( \frac{x}{2} \right) = \frac{1}{1 + \left(\frac{x}{2}\right)^2} \frac{1}{2} = \frac{1}{2 \left(1 + \left(\frac{x}{2}\right)^2\right)}.
\]

It follows that

\[
\int \frac{1}{4 + x^2} \, dx = \frac{1}{2} \arctan \left( \frac{x}{2} \right) + C.
\]

(e) \( \int e^x \cos(e^x) \, dx \).

**Solution:** We guess the form \( \sin(e^x) \) to get

\[
\frac{d}{dx} \sin(e^x) = \cos(e^x)e^x = e^x \cos(e^x).
\]

It follows that

\[
\int e^x \cos(e^x) \, dx = \sin(e^x) + C.
\]

2. Verify that the Riemann sum for \( f(x) = -x^2 + 2x \) evaluated between \( x = 0 \) and \( x = 2 \) is the same using the left and right endpoint of each rectangle to determine the height.

**Solution:** We have \( \Delta x = 2/n \) for both cases. Using the left endpoint
\[ x^n_i = a + (i - 1)\Delta x = 0 + 2(i - 1)/n = 2(i - 1)/n \text{ which gives} \]

\[
\lim_{n \to \infty} \sum_{i=1}^{n} f(x^n_i)\Delta x = \lim_{n \to \infty} \sum_{i=1}^{n} \left[ - \left( \frac{2(i - 1)}{n} \right)^2 + 2 \left( \frac{2(i - 1)}{n} \right) \right] \frac{2}{n}
\]

\[
= \lim_{n \to \infty} \sum_{i=1}^{n} \left[ - \frac{2(i - 1)^2}{n^2} + \frac{4(i - 1)}{n} \right] \frac{2}{n}
\]

\[
= \lim_{n \to \infty} \left[ - \frac{8}{n^3} \sum_{i=1}^{n} i^2 + \left( \frac{8}{n^2} + \frac{16}{n^3} \right) \sum_{i=1}^{n} i - \left( \frac{8}{n^2} + \frac{8}{n^3} \right) \sum_{i=1}^{n} 1 \right]
\]

\[
= \lim_{n \to \infty} \left[ - \frac{8}{n^3} \sum_{i=1}^{n} i^2 + \frac{8}{n^2} \sum_{i=1}^{n} i - \frac{8}{n^2} + \frac{8}{n^3} \right]
\]

\[
= - \frac{8}{3} + 4 = \frac{4}{3}.
\]

Using the right endpoint \( x^n_i = a + i\Delta x = 0 + 2i/n = 2i/n \) we have

\[
\lim_{n \to \infty} \sum_{i=1}^{n} f(x^n_i)\Delta x = \lim_{n \to \infty} \sum_{i=1}^{n} \left[ - \left( \frac{2i}{n} \right)^2 + 2 \left( \frac{2i}{n} \right) \right] \frac{2}{n}
\]

\[
= \lim_{n \to \infty} \left[ - \frac{8}{n^3} \sum_{i=1}^{n} i^2 + \frac{8}{n^2} \sum_{i=1}^{n} i \right]
\]

\[
= \lim_{n \to \infty} \left[ - \frac{8(n+1)(2n+1)}{6n^2} + \frac{8(n+1)}{2n} \right]
\]

\[
= - \frac{8}{3} + 4 = \frac{4}{3}.
\]

3. Truncated Riemann sums can be used to approximate areas even when the limit as \( n \to \infty \) cannot be easily evaluated.

(a) Use the first four Riemann rectangles to approximate the area under the curve \( f(x) = e^x \) bound between \( x = 0 \) and \( x = 1 \). Use both the left and right endpoints. Can we bound the actual area under the curve based on this information? Why or why not? [Hint: Consider the graph!]

Solution: We have

\[
\Delta x = \frac{b - a}{n} = \frac{1}{4}.
\]
Using the left end-points we have

\[
\text{Area} \approx \sum_{i=1}^{n} f(a + (i - 1)\Delta x)\Delta x
\]
\[
= \sum_{i=1}^{4} \frac{e^{(i-1)/4}}{4}
\]
\[
= \frac{1}{4} \left[ e^{0} + e^{1/4} + e^{1/2} + e^{3/4} \right]
\]
\[
\approx 1.51244.
\]

Using the left end-points we have

\[
\text{Area} \approx \sum_{i=1}^{n} f(a + i\Delta x)\Delta x
\]
\[
= \sum_{i=1}^{4} \frac{e^{i/4}}{4}
\]
\[
= \frac{1}{4} \left[ e^{1/4} + e^{1/2} + e^{3/4} + e \right]
\]
\[
\approx 1.94201.
\]

Since \( e^x \) is monotonically increasing, the rectangles with the height given by the left end-points consistently underestimate the actually area, while the rectangles with the height given by the right end-points consistently overestimate the actually area. It follows that

\[
1.51244 \leq \text{Area} \leq 1.94201.
\]

(b) Use the first four Riemann rectangles to approximate the area under the curve \( f(x) = \sin^2(x) \) bound between \( x = 0 \) and \( x = \pi \). Use both the left and right endpoints. Can we bound the actual area under the curve based on this information? Why or why not?

Solution: We have

\[
\Delta x = \frac{b - a}{n} = \frac{\pi - 0}{4} = \frac{\pi}{4}.
\]
Using the left end-points we have

\[
\text{Area} \approx \sum_{i=1}^{n} f(a + (i - 1)\Delta x)\Delta x
\]

\[
= \sum_{i=1}^{4} \frac{\pi \sin^2((i - 1)\pi/4)}{4}
\]

\[
= \frac{\pi}{4} \left[ \sin^2(0) + \sin^2\left(\frac{\pi}{4}\right) + \sin^2\left(\frac{\pi}{2}\right) + \sin^2\left(\frac{3\pi}{4}\right) \right]
\]

\[
= \frac{\pi}{4} \left[ (0)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + (1)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2 \right]
\]

\[
= \frac{\pi}{2}.
\]

Using the right end-points we have

\[
\text{Area} \approx \sum_{i=1}^{n} f(a + i\Delta x)\Delta x
\]

\[
= \sum_{i=1}^{4} \frac{\pi \sin^2(i\pi/4)}{4}
\]

\[
= \frac{\pi}{4} \left[ \sin^2\left(\frac{\pi}{4}\right) + \sin^2\left(\frac{\pi}{2}\right) + \sin^2\left(\frac{3\pi}{4}\right) + \sin^2(\pi) \right]
\]

\[
= \frac{\pi}{4} \left[ \left(\frac{1}{\sqrt{2}}\right)^2 + (1)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2 + (0)^2 \right]
\]

\[
= \frac{\pi}{2}.
\]

We might be tempted to say that the area must be \(\pi/2\). In fact, this is not justified because the function is not monotonically increasing or decreasing in the interval \(x = \pi\) to \(x = 2\pi\).

4. Jack has been on the road for half an hour, but his odometer is broken so he has no idea how far he has travelled. He has, however, been conscientious enough to look at his speedometer several times over the period and has jotted down the following values:

Based on what you know about the Riemann sum as an estimate of the area under a curve, find an estimate of how far Jack has travelled.

Solution: We know that velocity \(v(t)\) is the derivative of position \(p(x)\) (i.e. \(p'(x) = v(x)\)). That is to say, the position is the antiderivative of the velocity. Since we know antidifferentiation and computing
<table>
<thead>
<tr>
<th>Time (minutes)</th>
<th>0</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>Velocity (km/h)</td>
<td>70</td>
<td>75</td>
<td>85</td>
<td>80</td>
<td>100</td>
<td>90</td>
<td>85</td>
</tr>
</tbody>
</table>

Riemann sums is the same process by the Fundamental Theorem of Calculus, it follows that the position can roughly be given by the sum of the areas of the rectangles below the given points. We divide the region into six regions of width 5 minutes. Converting this into hours, we have 5 minutes = 1/12 hour. If we use the left end-points to determine the height of the rectangles, we have

\[
\text{Distance travelled} \approx \frac{1}{12} [70 + 75 + 85 + 80 + 100 + 90] = 41.667 \text{ km}.
\]

Using the right end-points, we have

\[
\text{Distance travelled} \approx \frac{1}{12} [75 + 85 + 80 + 100 + 90 + 85] = 42.917 \text{ km}.
\]