University of Waterloo
Midterm Examination
MATH 116
Calculus 1 for Engineering

Instructor: Matthew Douglas Johnston
Date: Friday, June 15, 2012
Term: 1125
Number of exam pages: 9
(including cover page)

Directions
1. Write your name and ID number at the top of this page.
2. Answer the questions in the spaces provided, using the backs of pages for overflow or rough work.
3. NO CALCULATORS ALLOWED! (You should not, however, need a calculator.)
4. Show all your work required to obtain your answers.

Formulas

- $\sin 2\alpha = 2 \sin \alpha \cos \alpha $
- $\sin^2 \alpha = (1 - \cos 2\alpha)/2$
- $\cos^2 \alpha = (1 + \cos 2\alpha)/2$

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1. Determine whether the following relationships between $y$ and $x$ can be represented as functions $y = f(x)$. If so, state the functional relationship. If not, state a pair of points which violate the functional requirements.

(a) $2 - yx - x + y = 0$,

(b) $x = y^2 - 2y + 1$,

(c) $x - y^3 = 0$.

2. Find the domain and range of the following functions:

(a) $f(x) = \ln(|x|)$,

(b) $f(x) = 2\sqrt{x}$,

(c) $f(x) = -\sqrt{x^2 - 2x + 2}$,

(d) $f(x) = \frac{x^2 - 4x + 4}{x - 2}$.
3. An important classification on the symmetry of a function is whether it is odd or even.

(a) Classify the following functions as odd, even, or neither:

(i) \( f(x) = \cos(x - \pi/2) \)

(ii) \( f(u) = ue^{-u^2} \)

(iii) \( f(x) = \frac{e^x + e^{-x}}{2} \)

(iv) \( f(x) = \frac{e^x - e^{-x}}{2} \)

(b) Every function can be written in the form 

\[ f(x) = f_e(x) + f_o(x) \]

where \( f_e(x) \) and \( f_o(x) \) are uniquely chosen even and odd functions, respectively. Find the functions \( f_e(x) \) and \( f_o(x) \) for which \( f(x) = f_e(x) + f_o(x) \) for the following.

(i) \( f(x) = \arctan(x) + 1 \)

(ii) \( f(x) = e^x \)
4. Consider the function

\[ f(x) = \frac{a + bx}{c + dx} \]

where \(a, b, c,\) and \(d\) are real numbers such that \(ad - bc \neq 0\).

(a) Find the inverse \(f^{-1}(x)\) (keep \(a, b, c,\) and \(d\) undetermined).

(b) Use the result of part (a) to determine the inverses of the following functions:

(i) \(f(x) = \frac{2 - 3x}{x - 1}\)

(ii) \(f(x) = \frac{75 + 18x}{81 + 17x}\)

(c) The condition \(ad - bc = 0\) characterizes the set of parameter values for which \(f(x)\) is non-invertible. What happens to \(f(x)\) which makes the function non-invertible for these values? (Hint: You may consider a representative set of values, e.g. \(a = b = c = d = 1\).)
5. Evaluate the following limits directly:

(a) \( \lim_{x \to 1^-} \frac{x^2 - 3x + 2}{|x - 1|} \)

(b) \( \lim_{x \to \infty} \sqrt{4x^2 - 3x + 1} - 2x \)

(c) \( \lim_{x \to \infty} \frac{\sin^2(x)}{x^3 - 1} \)

(d) \( \lim_{x \to \infty} \frac{3^x - 1}{3^{x+1} + 2^x} \)
6. Use the formal definition of a derivative to find $f'(x)$ for the following:

(a) $f(x) = \frac{1}{1+x}$

(b) $f(x) = |x|, \quad x \neq 0$

7. Determine $\frac{dy}{dx}$ given the following relationship:

$$\frac{x - y}{x + y} = 1.$$
8. The number of cells in a culture of bacteria is given as a function of time by the formula

\[ P(t) = \frac{K P_0 e^{rt}}{K + P_0 (e^{rt} - 1)} \]

where \( K \) is the carrying capacity of the environment, \( r \) is a scale constant, and \( P_0 \) is the initial population size.

(a) Find the inverse \( P^{-1}(t) \). Briefly explain what this dependence means.

(b) Assume that \( r > 0 \). Determine the “long-term” behaviour of \( P(t) \); that is to say, evaluate \( \lim_{t \to \infty} P(t) \). Briefly interpret this result.
9. Many switch-like mechanisms are modelled using the Heaviside step function

\[ H(x) = \begin{cases} 
0, & \text{for } x < 0 \\
1, & \text{for } x \geq 0.
\end{cases} \]

(a) Write \( f(x) = |2x - x^2| \) using Heaviside functions.

(b) Write \( f(x) = H(x + 2) + H(x + 1) - H(x - 1) - H(x - 2) \) as a piecewise defined function.

(c) [Bonus!] Graph \( f(x) = H(1 - x^2) \). [Hint: Be careful to consider the endpoints of each interval!]

THIS PAGE IS FOR ROUGH WORK