1 Partial fractions

Many applications require us to work with rational functions, i.e. functions of the form

\[ f(x) = \frac{p(x)}{q(x)} \]

where \( p(x) \) and \( q(x) \) are polynomials.

In general, if we add several rational functions together, we bring them together by finding a common denominator. For example, we know

\[ \frac{1}{x - 1} + \frac{1}{x + 1} = \frac{(x + 1) + (x - 1)}{(x - 1)(x + 1)} = \frac{2x}{x^2 - 1}. \]

In many ways, this is a simpler form, since there is only a single term; however, this simplified form is not the form preferred for all applications. Many applications in Calculus 2 (and second-year differential equations courses!) require us to go in the other direction, i.e. take the simplified form with the common denominator and expand it out.

The question then becomes, given the form

\[ \frac{2x}{x^2 - 1}, \]

how do we recover the expanded form

\[ \frac{1}{x - 1} + \frac{1}{x + 1}? \]

It turns out that there is a general method for doing this. It is called partial fraction decomposition.

Roughly, we assume that a form of the desired type is possible and go from there. That is to say, we assume there is a partial fraction decomposition

\[ \frac{2x}{x^2 - 1} = \frac{A}{x - 1} + \frac{B}{x + 1} \]
and solve for $A$ and $B$. For this example, we have
\[
\frac{2x}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}
\implies 2x = A(x+1) + B(x-1)
\]
This needs to hold for all values of $x$, so we can pick any two values we want to solve for $A$ and $B$. The easiest choices are $x = 1$ and $x = -1$ which give
\[
x = 1 \implies 2 = 2A \implies A = 1
\]
\[
x = -1 \implies -2 = -2B \implies B = 1.
\]
It follows that
\[
\frac{2x}{x^2 - 1} = \frac{1}{x-1} + \frac{1}{x+1}.
\]
That is all well and good for this example, but there are obviously more complicated things that can happen in general. Consider the the general rational function
\[
f(x) = \frac{p(x)}{q(x)}
\]
where $p(x)$ and $q(x)$ are polynomials, i.e. they have the form $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ and $q(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0$.

The algorithm for partial fraction decomposition is given by the following:

1. If the order of $q(x)$ is greater than the order of $p(x)$ (i.e. $m > n$), move on to step two. If this is not the case, we have to perform polynomial long division. We will assume from now on that $q(x)$ is higher order than $p(x)$.

2. Fully factor $q(x)$. The Fundamental Theorem of Algebra (mathematicians love fundamental theorems!) guarantees that any polynomial can be factored uniquely into chains of terms of one of two forms:
\[
(ax + b)^n \quad \text{or} \quad (ax^2 + bx + c)^n.
\]

3. For every terms of the form $(ax + b)^n$ in $q(x)$ add
\[
\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}
\]
(1)
to the right-hand side. For every term of the form $(ax^2 + bx + c)^n$ add
\[
\frac{B_1 x + C_1}{ax^2 + bx + c} + \frac{B_2 x + C_2}{(ax^2 + bx + c)^2} + \cdots + \frac{B_n x + C_n}{(ax^2 + bx + c)^n}.
\]
(2)
4. Multiply across by the factored form of \( q(x) \), cancelling terms where appropriate.

5. Expand both sides and order in terms of powers of \( x \). Equate the coefficients of the corresponding powers on the left-hand and right-hand side. Solve the resulting linear system of equations.

**Example 1:** Set up the partial fraction expansion of

\[
\frac{1}{(x - 1)(2x + 3)^3(x^2 + x + 1)^2(3x^2 - x + 3)}.
\]

Do not evaluate for the constants!

**Solution:** This is a straightforward application of (1) and (2). We have

\[
\frac{1}{(x - 1)(2x + 3)^3(x^2 + x + 1)^2(3x^2 - x + 3)} = \frac{A}{x - 1} + \frac{B}{2x + 3} + \frac{C}{(2x + 3)^2} + \frac{D}{(2x + 3)^3} + \frac{Ex + F}{x^2 + x + 1} + \frac{Gx + H}{(x^2 + x + 1)^2} + \frac{Ix + J}{3x^2 - x + 3}.
\]

**Example 2:** Find the partial fraction decomposition of

\[
\frac{x}{(x - 1)(x + 1)(x + 3)}.
\]

**Solution:** The order of the denominator is clearly higher than the numerator, so we can skip step 1. The denominator is already factored, so we can also skip step 2. All the factored terms fit the form of (1) so we have

\[
\frac{x}{(x - 1)(x + 1)(x + 3)} = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{x + 3}.
\]

We can multiply across by the denominator on the left-hand side to get

\[
x = A(x + 1)(x + 3) + B(x - 1)(x + 3) + C(x - 1)(x + 1).
\]

There are several ways to solve for \( A, B, \) and \( C \). We notice first of all that, in order for (3) to be satisfied, it must be satisfied for all \( x \). This means that we can select any value of \( x \) we want to solve for the constants \( A, B \) and \( C \)!
Which values of $x$ should we choose? We notice the brackets on the right-hand side have zeroes at the values $x = 1$, $x = -1$, and $x = -3$. It will simplify our algebra to use these values.

Setting $x = 1$ we obtain $1 = A(2)(6)$ which implies $A = \frac{1}{8}$.

Setting $x = -1$ we obtain $-1 = B(-2)(2)$ which implies $B = \frac{1}{4}$.

Setting $x = -3$ we obtain $3 = C(-4)(-2)$ which implies $C = -\frac{3}{8}$.

That was not nearly as painful as it could have been, but let’s keep in mind what we have actually done. This means that

$$\frac{x}{(x-1)(x+1)(x+3)} = \frac{1}{8(x-1)} + \frac{1}{4(x+1)} - \frac{3}{8(x+3)}$$

(Notice that we can check to see if this answer is valid—i.e. whether the process is valid—by simply finding a common denominator. The process is tedious, but we might as well go through it once to convince ourselves that our method works:

$$\begin{align*}
\frac{1}{8(x-1)} + \frac{1}{4(x+1)} - \frac{3}{8(x+3)} & = \frac{1}{4} \left[ \frac{1}{2(x-1)} + \frac{1}{x+1} - \frac{3}{2(x+3)} \right] \\
& = \frac{1}{4} \left[ \frac{(x+1)(x+3) + 2(x-1)(x+3) - 3(x+1)(x-1)}{2(x-1)(x+1)(x+3)} \right] \\
& = \frac{1}{8} \left[ \frac{x^2 + 4x + 3 + 2x^2 + 4x - 6 - 3x^2 + 3}{(x-1)(x+1)(x+3)} \right] \\
& = \frac{x}{(x-1)(x+1)(x+3)}.
\end{align*}$$

So our partial fraction decomposition checks out!)

**Example 3:** Find the partial fraction decomposition of

$$\frac{x^4 + x^3 + x^2 - x}{x^3 - 1}.$$

**Solution:** We notice first of that the order of the numerator is higher than the denominator, so we need to perform long division. We can perform standard long division or synthetic division (for those who know it). We can also notice that we can factor

$$x^4 + x^3 + x^2 - x = x(x^3 - 1) + (x^3 - 1) + x^2 + 1 = (x + 1)(x^3 - 1) + x^2 + 1.$$
This implies immediately that
\[
\frac{x^4 + x^3 + x^2 - x}{x^3 - 1} = \frac{(x + 1)(x^3 - 1) + x^2 + 1}{x^3 - 1} = x + 1 + \frac{x^2 + 1}{x^3 - 1}.
\]

In order to perform a partial fraction decomposition we need to be able to factor \(x^3 - 1\) into a chain of terms of form (1) or (2). The decomposition is given by the difference of cubes formula

\[x^3 - 1 = (x - 1)(x^2 + x + 1)\]

where \(x^2 + x + 1\) cannot be further decomposed.

We set-up our partial fraction decomposition for three variables \(A, B,\) and \(C\) so that

\[\frac{x^2 + 1}{(x - 1)(x^2 + x + 1)} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + x + 1}.\]

Multiplying across by the denominator on the left-hand side we arrive at the more manageable form

\[x^2 + 1 = A(x^2 + x + 1) + (Bx + C)(x - 1).\]

We recall that in order for our partial fraction decomposition to be valid, the above expression must hold for all values of \(x\). This suggests two alternative methods of solving for the constants.

1. Plug values of \(x\) into the equation until you have enough expressions to solve for the variables. Particularly useful values of \(x\) are those which eliminate brackets (e.g. if \((x - 2)\) appears factored several times, select \(x = 2\)).

2. Collect powers of \(x\) on the right-hand side and then equate coefficients on the left-hand and right-hand side.

For illustrative purposes, we will perform both methods here.

To the first method, we select the values \(x = 0, x = 1\) and \(x = -1\). Plugging \(x = 0\) into the expression, we have \(1 = A - C\) which implies \(C = A - 1\). Plugging in \(x = 1\) gives \(2 = 3A\) which implies \(A = \frac{2}{3}\), and
therefore that $C = -\frac{1}{3}$. Plugging in $x = -1$ gives $2 = A - 2C + 2B$. We can plug in our known values of $A$ and $C$ and solve for $B$ to get $B = \frac{1}{3}$.

Alternative, we can expand our original expression to get

$$x^2 + 1 = (A + B)x^2 + (A - B + C)x + (A - C).$$

Equating the coefficients of the $x$ terms on the left-hand and right-hand side (realizing that $x^2 + 1 = (1)x^2 + (0)x + (1)$) gives the system of equations

$$\begin{align*}
A + B &= 1 \\
A - B + C &= 0 \\
A - C &= 1.
\end{align*}$$

For those who are familiar with matrix analysis, this can be solved through row reduction. Otherwise, we back substitute variables to get

$$C = A - \frac{1}{3} \Rightarrow A - B + C = 2A - B = 1 \Rightarrow B = 2A - 1 \Rightarrow A + B = 3A = 2 \Rightarrow A = \frac{2}{3} \Rightarrow B = \frac{1}{3} \Rightarrow C - \frac{1}{3}.$$  

We put this together to get

$$\frac{x^2 + 1}{(x - 1)(x^2 + x + 1)} = \frac{2}{3(x - 1)} + \frac{x - 1}{3(x^2 + x + 1)},$$

so that

$$\frac{x^4 + x^3 + x^2 - x}{x^3 - 1} = x + 1 + \frac{2}{3(x - 1)} + \frac{x - 1}{3(x^2 + x + 1)}.$$ 

**Example 4:** Find the partial fraction decomposition of

$$\frac{-2x^2 - 4x + 1}{4x^4 - 4x^2 + 1}.$$ 

**Solution:** We notice that the denominator is higher-order than the numerator, so we do not need to worry about long-division. Our first step is to factor the denominator. We notice by difference of squares that

$$4x^4 - 4x^2 + 1 = (2x^2 - 1)^2.$$ 

We could factor this further by taking $(2x^2 - 1) = (\sqrt{2}x + 1)(\sqrt{2}x - 1)$; however, this will lead to more work than doing our partial fraction decomposition on this term directly. You can try the extra expansion if you like.
We consider the partial fraction decomposition

\[
\frac{-2x^2 - 4x + 1}{(2x^2 - 1)^2} = \frac{Ax - B}{2x^2 - 1} + \frac{Cx - D}{(2x^2 - 1)^2}.
\]

We rearrange this so that

\[
-2x^2 - 4x + 1 = (Ax - B)(2x^2 - 1) + Cx + D
\]

\[
= Ax^3 + 2Bx^2 + [C - A]x + [D - B].
\]

We equate coefficients on the left- and right-hand sides to get \( A = 0, B = -1, C = -4, \) and \( D = 0. \) This leads to

\[
\frac{-2x^2 - 4x + 1}{(2x^2 - 1)^2} = \frac{1}{1 - 2x^2} - \frac{4x}{(2x^2 - 1)^2}.
\]