1 Continuity

One of the principal properties of a function which we will be interested in is continuity.

Continuity is basically a measure of how connected a function is. That is to say, if a function is continuous, it cannot have any disconnections, or jumps. If we imagine drawing the function on a piece of paper, we must be able to draw the function without lifting the pencil from the paper.

There are several mathematical definitions of what it means to be continuous (including an $\epsilon$-$\delta$ definition!). To keep the mathematical rigor at a minimum, we will use the following equivalent (and less cumbersome) definition.

**Definition 1.1.** A function $f(x)$ is **continuous at a point** $a$ if and only if

$$
\lim_{x \to a^-} f(x) = f(a) = \lim_{x \to a^+} f(x).
$$

That is to say, a function is continuous at a point $a$ if the limit from the left equals the limit from the right, and they both equal the defined value at the point.

There are several things to note about continuity:

1. If a function is not continuous at a point $x = a$, we will say it is **discontinuous** at $a$.

2. If a function is continuous at all points $x \in D(f)$, we will say $f(x)$ is **continuous**.

3. Just because a function is continuous does **not** mean that it is smooth. A function can have sharp points and still be continuous (e.g. $f(x) = |x|$ is continuous but pointed at $x = 0$).

There are several common forms of discontinuities:

1. **Asymptotes:** If we have $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$, the function cannot be continuous at $a$. 


2. **Jump discontinuities:** If \( \lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x) \) the function cannot be continuous at \( a \) and this kind of discontinuity is called a *jump* discontinuity.

3. **Removable discontinuities:** If \( \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) \) but this does not equal \( f(a) \) we have what is called a *removable* discontinuity. These are so named because, of the discontinuities we have discussed, they are the closest to being continuous—all we have to do is redefine the value at \( f(a) \)!

There are several nice properties of continuous functions. Suppose that \( f(x) \) and \( g(x) \) are continuous. Then:

1. \( f(x) \pm g(x) \) is continuous.
2. \( f(x) \cdot g(x) \) is continuous.
3. \( f(x)/g(x) \) is continuous at all points such that \( g(x) \neq 0 \).

**Example 1:** State any points \( x \in \mathbb{R} \) at which the following function is discontinuous. State the kind of each discontinuity.

\[
f(x) = \begin{cases} 
x, & \text{for } x < -1 \\
\frac{1}{x}, & \text{for } -1 \leq x < 1 \\
\frac{(x-1)^2}{x} + 3, & \text{for } x \geq 1. 
\end{cases}
\]

**Solution:** We know that \( x \) is continuous so that \( f(x) \) is continuous on \( x < -1 \). For continuity at \( x = -1 \) we check

\[
\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} x = -1
\]

and

\[
\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} \frac{1}{x} = -1.
\]

Since these equal, \( f(x) \) is continuous at \( x = -1 \).

In the interval \(-1 < x < 1\), we notice that \( 1/x \) has a discontinuity at \( x = 0 \). This discontinuity is an *asymptote*.

At \( x = 1 \), we check

\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{1}{x} = 1
\]

and

\[
\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x - 1)^2 + 3 = 3.
\]
Since these do not equal, \( f(x) \) is discontinuous at \( x = 1 \). This is a \textit{jump discontinuity}.

Finally, we know that \((x - 1)^2 + 3\) is continuous, so there are no further discontinuities.

## 2 Intermediate Value Theorem

An important property of continuous functions is that, given two values on the curve, the function must go through all \textit{intermediate} values. This is a simple consequence of the fact that continuous functions cannot, under any circumstance, jump from one value to another.

This is stated mathematically as the following.

\textbf{Theorem 2.1} (Intermediate Value Theorem). Suppose \( f(x) \) is continuous on the domain \([a, b]\). Then, for any value \( u \) between \( f(a) \) and \( f(b) \) there is a value \( c \in (a, b) \) such that \( f(c) = u \).

\textbf{Note}: The Intermediate Value Theorem (shortformed IVT) is most commonly used to determine the roots of functions. We are often interested in the roots of functions (for reasons we will see shortly) and cannot always evaluate them. But, if \( f(a) \) and \( f(b) \) take opposite signs (i.e. one positive, one negative) the IVT guarantees a root in the interval \((a, b)\)!

\textbf{Example 1}: Show that \( e^x \) and \( x \) intersect in the interval \([0, 1]\).

\textbf{Solution}: Finding where \( e^{-x} \) and \( x \) intersect is equivalent to solving \( e^x = x \). We cannot explicitly solve for \( x \) so we are left with proving there is an intersection point, and asking, if there is, where (roughly) it happens to be. How might we proceed?

Well, \( e^{-x} = x \) is equivalent to \( e^{-x} - x = 0 \). In other words, finding the points where \( e^{-x} \) and \( x \) intersect is the same as finding the roots of the function \( f(x) = e^{-x} - x \). We evaluate at the endpoints of the interval \([0, 1]\) to get

\[ f(0) = 1, \quad \text{and} \quad f(1) = e^{-1} - 1 \approx -0.632 < 0. \]

It follows immediately by the IVP there is a point \( c \in (0, 1) \) where \( f(c) = 0 \) (since 0 lies between \( f(0) = 1 \) and \( f(1) \approx -0.632 \) and \( f(x) \) is continuous). We are done!