1 Implicit Differentiation

There are other applications of the chain rule which we will go over now.

Consider the question of determining the slope of the tangent line to the unit circle centred at (0, 0), at the point (4/5, 3/5). We can see immediately from the graph that there are two positive points on the curve and that one has negative slope and the other has positive slope (see Figure 1).

The question remains, however, as to how we find the derivative analytically. We have the equation of the unit circle given by $x^2 + y^2 = 1$. Our traditional method for finding a derivative is solving for $y$ as a function of $x$ (i.e. $y = f(x)$) and then taking the derivative of $f(x)$. This intuition does not help us here because we cannot solve for $y$ as a function of $x$!

The problem is that we cannot write $y$ explicitly as a function of $x$—rather, the variables are related implicitly. So how do we take the derivative? The chain rule comes to our rescue!

Figure 1: There are two positive tangent lines to points with $x = 4/5$, one with negative slope, one with positive slope.

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The problem is that we cannot write $y$ explicitly as a function of $x$—rather, the variables are related implicitly. So how do we take the derivative? The chain rule comes to our rescue!
Even though we cannot solve for $y$ in terms of $x$ explicitly, we can treat $y$ as a function of $x$, $y = f(x)$, and take the derivative of both sides with respect to $x$. We have that
\[ \frac{d}{dx} x^2 = 2x \]
and
\[ \frac{d}{dx} 1 = 0 \]
but how do we handle the term $y^2$? Well, consider the replacement $y = f(x)$. This gives $y^2 = f(x)^2$. We can evaluate this by using the chain rule! We have
\[ \frac{d}{dx} y^2 = \frac{d}{dx} f(x)^2 = 2f(x) \left[ \frac{d}{dx} f(x) \right] = 2f(x)f'(x) = 2yy'. \]
In other words, we can differentiate both sides of $x^2 + y^2 = 1$ with respect to $x$ to get
\[ 2x + 2yy' = 0. \]
Because of how $y'$ fell out of $y^2$ by the chain rule, we can solve for it to get
\[ \frac{dy}{dx} = \frac{-x}{y}. \]
In other words, even though we could not solve for $y$ in terms of a function of $x$ explicitly, we can solve for $y'$! We can see immediately that, plugging in the point $(4/5, 3/5)$ we have $y' = 4/3$, and plugging in the point $(4/5, -3/5)$ we have $y' = -4/3$ (see Figure 1).

There are several things to note:

1. Although this is an application of the chain rule, it is commonly given its own name. It is called implicit differentiation.

2. Again, although this is an application of the chain, it is more common to consider implicit differentiation as its own operation. The general procedure is to take every term with a $y$, differentiate as though the $y$ were an $x$, and then multiply by $y'$.

3. Because $x$ are now permitted to give multiple $y$ values, in general we will need to solve for $y'$ and then given both an $x$ and a $y$ value to determine the derivative at a point.
Example 1: Find the derivative $y'$ at $(0,0)$ of the following relation:

$$\sin(xy) + y = x.$$ 

Solution: We can implicitly differentiate both sides with respect to $x$ to get

$$\cos(xy)(y + xy') + y' = 1$$

$$\Rightarrow \cos(xy)xy' + y' = 1 - \cos(xy)y$$

$$\Rightarrow \frac{dy}{dx} = \frac{1 - y\cos(xy)}{1 + x\cos(xy)}.$$ 

At $(0,0)$ we have

$$\frac{dy}{dx} = \frac{1 - (0)\cos(0)}{1 + (0)\cos(0)} = 1.$$ 

Example 2: Find the points where $y' = 0$ for the following relationship:

$$x^2 + xy + y^2 = 1.$$ 

Solution: We have

$$2x + y + xy' + 2yy' = 0$$

$$\Rightarrow (x + 2y)y' = -2x - y$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x + y}{x + 2y} = 0.$$ 

In order to have this equal to zero we need to have $y = -2x$. Along this line, $y' = 0$, but it is not true that every point on $y = -2x$ satisfies the original relationship $x^2 + xy + y^2 = 1$. In order to find the points which satisfy both of these requirements, we substitute $y = -2x$ into the relationship to get

$$x^2 + x(-2x) + (-2x)^2 = 1$$

$$\Rightarrow x^2 - 2x^2 + 4x^2 = 1$$

$$\Rightarrow 3x^2 = 1 \Rightarrow x = \pm \frac{1}{\sqrt{3}}.$$ 

We can substitute this into $y = -2x$ to get the points $(1/\sqrt{3}, -2/\sqrt{3})$ and $(-1/\sqrt{3}, 2/\sqrt{3})$. (See Figure 2).
Figure 2: Implicit plot of $x^2 + xy + y^2 = 1$. 