1 Local and Global Extrema

We are often interested in the maximal and minimal value of things.

Suppose that we know are running a toy store and are trying to determine a reasonable price for our toy. We know that a low price will generate strong sales but at a small profit margin per unit, while a higher price will discourage sales but generate strong per unit profit. We expect to generate strongly profit in the ground in between these extremes. How do we quantify this?

Suppose we let $x$ determine the per unit cost in dollars. Suppose each toy costs $2 to manufacture. Suppose we know that the demand decreases linearly according to $1000 - 25x$ as the price increases. That is to say, for each dollar we increase the cost, we expect to lose 25 customers. Suppose further that the company’s board has told us we may not sell the toy for less than $5 or more than $30.

Now we ask the questions:

(1) At what price do we maximize our profit?

(2) At what price do we minimize our profit?

Let’s set up the question mathematically. The profit per unit is given by the selling cost minus the production cost $x - 2$; the overall profit is given by this times the demand, which is


We want to maximize this over the domain $x \in [5, 30]$. How do we approach this problem?

We know that the function $P(x)$ is a parabola opening down so that the vertex must be the highest point. Let’s take an alternative approach. We know that in order for a function to have a maximum it must increase (i.e. $f'(x) > 0$) and then decrease (i.e. $f'(x) < 0$). It is intuitive to suppose that the maximum occurs at the critical point where $f'(x) = 0$! We can see from the graph that this intuition is justified (see Figure 1). We have

$$f'(x) = -50x + 1050 = 0 \implies x = 21.$$
At the value $x = 21$ we have $P(21) = 9025$. This is the maximal value!

Figure 1: The profit function $P(x) = (x-2)(1000-25x)$. The global maximum is $(21, 9025)$. The global minimum is $(5, 2625)$. The point $(30, 7000)$ is a local minimum.

To answer the question of which price minimizes profit, we notice that there are no corresponding critical points. Instead, we must look at the endpoints of the interval. We can see that $P(x)$ is lower at $x = 5$ ($P(5) = 2625$) than $x = 30$ ($P(30) = 7000$) so that $(5, 2625)$ is the global minimum, while $(30, 7000)$ is a local minimum only.

We notice several things about this process which are worth emphasizing:

1. A function $f(x)$ has a **global maximum** (respectively, **minimum**) at $x^*$ if $f(x^*)$ is greater than (less than) $f(x)$ for every other value $x$ in the domain of $f$.

2. A function $f(x)$ has a **local maximum** (respectively, **minimum**) at $x^*$ is $f(x^*)$ is greater than (less than) $f(x)$ for every other value $x$ in a small neighbourhood of $x^*$.

3. A function $f(x)$ may attain a maximum or minimum at $x^*$ if and only if $x^*$ is a critical point or an endpoint of the domain of $f$.

4. A critical point $x^*$ is a maximum (respectively, minimum) if $f'(x) > 0$ ($f'(x) < 0$) for $x^* - \epsilon < x < x^*$ and $f'(x) < 0$ ($f'(x) > 0$) for $x^* < x < x^* + \epsilon$ for some small $\epsilon > 0$.

5. If a critical point $x^*$ for which $f'(x) > 0$ (respectively, $f'(x) < 0$) for $x^* - \epsilon < x < x^*$ and $f'(x) > 0$ ($f'(x) < 0$) for $x^* < x < x^* + \epsilon$ for some small $\epsilon > 0$, $x^*$ is neither a maxima nor a minima.
By point (2), in order to determine the global and local extrema of a continuously differentiable function \( f(x) \), it is sufficient to find all the critical points and evaluate \( f(x) \) at them and the endpoints!

**Example 1:** Find and classify the extrema of \( f(x) = x^3 \) on the interval \([-1,1]\).

**Solution:** We know that extrema can only happen only at critical points and endpoints, so we find

\[
f'(x) = 3x^2 = 0 \implies x = 0.
\]

We have a single critical point, \( x = 0 \), and two endpoints \( x = -1 \) and \( x = 1 \). We have

\[
f(-1) = -1 \\
f(0) = 0 \\
f(1) = 1.
\]

It follows that \( f(-1) = -1 \) is the global minimum and that \( f(1) = 1 \) is the global maximum, but what about \( f(0) = 0 \)? We check the derivatives on either side of \( x = 0 \) to realize that we have \( f'(x) > 0 \) no matter side we are on. It follows that \( x = 0 \) corresponds to neither a local maximum nor a local minimum.

**Example 2:** Find and classify the extrema of \( f(x) = x^3 - 9x^2 + 24x - 16 \) on the interval \([0,5]\).

**Solution:** We check the critical points to get

\[
f'(x) = 3x^2 - 18x + 24 = 3(x^2 - 6x + 8) = 3(x - 4)(x - 2) = 0.
\]

It follows that the critical points are \( x = 2 \) and \( x = 4 \). We need to check the value of the function at the critical points, \( x = 2 \) and \( x = 4 \), and the endpoints, \( x = 0 \) and \( x = 5 \). We have

\[
f(0) = -16 \\
f(2) = 4 \\
f(4) = 0 \\
f(5) = 4.
\]
We can see that the global minimum is $-16$ and occurs at $x = 0$, and that
the global maximum is $4$ and is attained at $x = 2$ and $x = 5$. To classify
the point $f(4)$ we notice that $f'(x) < 0$ for $2 < x < 4$ and $f'(x) > 0$ for
$4 < x < 5$. It follows that $x = 4$ corresponds to a global minimum.