1 Optimization

Consider the problem of enclosing a yard connecting to the back wall of a school. Suppose we have 500 meters of fencing and want to enclose the biggest rectangular yard we possibly can. How might we approach this problem?

Consider the picture given in Figure 1. If we assign $x$ to be the length of the side adjacent to the school, and $y$ to be the side opposite, the perimeter is given by

$$P = 2x + y.$$ 

This is fixed to be 500 so that $2x + y = 500$, which implies $y = 500 - 2x$. The area is given by the formula

$$A = xy$$

which, in terms of just $x$, can be stated as $A = x(500 - 2x) = -2x^2 + 500x$.

![Figure 1: Consider a yard built off the back of a school with sides $x$, $x$, and $y$. The perimeter is given by $2x + y$ and is fixed to be 500. We want to find the side lengths which maximize the area $xy$.](image-url)
What we want to do now is find the \( x \) value which maximizes this function. In order to do this, we rely on the properties of maxima and minima which we discussed last week. We know that maxima and minima can only occur at critical points and endpoints of our interval of interest. By inspection, our interval of interest is \( 0 \leq x \leq 250 \). To find the critical points, we take the derivative to get

\[
f'(x) = -4x + 500 = 0 \implies x = 125.
\]

We have \( f'(x) > 0 \) for \( 0 < x < 125 \) and \( f'(x) < 0 \) for \( 125 < x < 250 \) so that this is a maximum. It follows that the area is maximized when \( x = 125 \) m and \( y = 250 \) m, and the maximal area is \( 31250 \text{ m}^2 \).

This process is just the tip of the iceberg of the mathematical discipline of *optimization*. This is a very rich area of mathematics which has applications in disciplines as varied as urban planning, network analysis, probability theory, transportation problems, economics, and many more.

The general factors in an optimization problem are the following:

1. **Objective function** - This is what we are optimizing, i.e. the thing we are trying to maximize or minimize (e.g. maximizing profit, minimizing costs, maximizing areas, etc.).

2. **Constraints** - These are the relationships between the variables of the problem. (For purposes of this course, they will let us reduce the objective function to a single variable!)

   Once we have the objective function adequately stated in terms of a single variable (and know the domain!), we perform our earlier methods for finding local and global extrama of this function.

**Example 1:** Consider being asked to bury a cable and connect it from your house to a station 8 km down a straight road. Your house is situated 6 km directly in from the road and is surrounded by brush. The cost of burying the cable in the brush is $250/km while the cost of burying it along the road is $100/km. How should you bury the cable to minimize your costs?

**Solution:** There are two intuitive cases we might consider. We could simple move in a straight line from your house to the station. But then we are burying the cable entirely in the brush and incur the high associated cost of doing so. This is probably not optimal.

You could also move in a straight line from your house to the road and then straight along the road, but this would incur a high overall cost.
associated with making the cable unnecessarily long. This is also probably not optimal.

In reality, the optimal solution is probably somewhere in between these two extremes: we move at an angle in a straight line to the road, keeping the length of the cable relatively short, and then the rest of the way along the road, which keeps our per kilometer costs down. (See Figure 2.)

![Diagram of cable placement](image)

Figure 2: If we set $x$ to denote the meeting point at the road, we can use Pythagoras Theorem to determine the rest of the sides.

So how do we set this up as an optimization problem? What we want to determine is, as we vary the meeting point with the road how the total cost of burying the cable changes. We can complete the sides of the triangles according to that outlined in Figure 2. If we let $x$ denote the meeting point (relative to the home meeting the road perpendicularly), we can fill in all the other sides of our triangles in terms of $x$.

But what are we optimizing? For every kilometer traveled along the intermediate line in the triangle, we have an associated cost of $250$. For every kilometer travelled the rest of the way along the road, we have the associated cost of $100$. It follows that the cost function $C(x)$ is given by

$$C(x) = 250\sqrt{x^2 + 36} + (100)(8 - x)$$

$$= 250\sqrt{x^2 + 36} - 100x + 800.$$  

We needs to find the minimum of this relative to the domain $0 \leq x \leq 10$. We need to find the critical points. We have

$$C'(x) = 125\frac{2x}{\sqrt{x^2 + 36}} - 100$$

$$= \frac{250x}{\sqrt{x^2 + 36}} - 100 = 0.$$
We can rearrange this to get

\[ 250x = 100\sqrt{x^2 + 36}. \]

Squaring both sides gives

\[ 62500x^2 = 10000(x^2 + 36) \implies 52500x^2 = 360000 \implies x = \sqrt{\frac{48}{7}}. \]

This value is approximately equal to \( x \approx 2.6186 \). We also consider the endpoints \( x = 0 \) and \( x = 8 \). We have that

\[
\begin{align*}
C(0) &= 2300 \\
C(2.6186) &\approx 2174 \\
C(8) &= 2500.
\end{align*}
\]

It follows that, in order to minimize the costs associated with burying our cable, we need to pick the point 2.6186 km down the road from our house.