1 Related Rates

Consider the following situation. Suppose we are filling up a spherical balloon with a pump that blows air into the balloon at a fixed rate of $4 \text{ L/min}$. How quickly is the surface area of the balloon increasing when the balloon has a radius of $8 \text{ cms}$?

This is a different type of question than the optimization questions we have been evaluating to this point. The primary difference is that both the volume and surface area are driven by the independent factor: time. That is to say, we are not asking how the volume and surface area related to one another, but how they relate to an independent variable.

So how do we handle a question like this? Let’s write down all of the information we have at our disposal:

(Volume) \[ V = \frac{4}{3} \pi r^3 \]

(Surface Area) \[ S = 4 \pi r^2 \]

\[ \frac{dV}{dt} = 4 \]

\[ \frac{dS}{dt} = ? \quad (\text{when } r = 8). \]

We can relate changes in volume to changes in the radius to changes in surface area in the following way. We take the derivative (implicitly) of the equations for both volume and surface area with respect to time to obtain

\[ \frac{dV}{dt} = 4 \pi r^2 \frac{dr}{dt} = 4 \implies \frac{dr}{dt} = \frac{1}{\pi r^2} \]

\[ \frac{dS}{dt} = 8 \pi r \frac{dr}{dt} = 8 \pi r \frac{1}{\pi r^2} = \frac{8}{r}. \]

It follows that when the radius is $8 \text{ cms}$ (i.e. $r = 8$) we have

\[ \frac{dS}{dt} = 1 \]

so that the surface area is increasing at a rate of $1 \text{ cm}^2/\text{s}$. 

1
A problem of this type is called a related rate problem. Related rates problems have a few of the following general features:

1. There is a set of variables which are undergoing change with respect to an independent variable. That is to say, there is an independent variable which is driving changes in several dependent variables.

2. The dependent variables undergoing change are related. That is to say, there is some algebraic way of relating one of the variables to the other. (In the volume to surface area question, we have two equations, which allows us to relate the variables through the radius.)

Although there is no general formula for solving related rates problems as such (each one is different!), a few hints are particularly helpful:

1. It is often most difficult to know where to begin the problem. When in doubt, list all of the given information in mathematical terms and ask what it is you want to find.

2. Draw a picture!

3. Be careful to take derivatives with respect to the driving variable. (In the relationship between volume, surface area, and the radius, we took all derivatives with respect to $t$.)

Example 1: Larry and Mary go on a date and leave the restaurant at the same time. Larry travels due east at a speed of 60 km/hr while Mary travels due south at a speed of 80 km/hr. How fast is the distance between them growing after they have been travelling for half an hour?

Solution: We draw the picture given in Figure 1. Not knowing a better way to start, we write out the given information:

\[
D^2 = x^2 + y^2
\]
\[
\frac{dx}{dt} = 60
\]
\[
\frac{dy}{dt} = 80
\]

(Larry’s distance travelled) \( x(0.5) = 30 \)

(Mary’s distance travelled) \( y(0.5) = 40 \)

(Distance between them) \( D(0.5) = 50 \)

\[
\frac{dD}{dt} = ?
\]

2
Figure 1: The picture associated with the problem described above.

The way to procedure should not be clear. If we differentiate the distance formula (with respect to time) we obtain

\[ 2D \frac{dD}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \implies \frac{dD}{dt} = \frac{x}{D} \frac{dx}{dt} + \frac{y}{D} \frac{dy}{dt}. \]

We have all the given information to solve for the right-hand side, so that

\[ \frac{dD}{dt} = \frac{30}{50}(60) + \frac{40}{50}(80) = 100. \]

It follows that, after half an hour of travelling, the distance between Larry and Mary is growing by 100 km/hr.

**Example 2:** Suppose we are tracking a boat which is travelling 30 km/h from north to south three kilometers off the coast of a straight shoreline. If we are situated right on the shore, how quickly (in radians) are we pivoting to track the boat when it is approaching and making a 45 degree angle with the shore.

**Solution:** We have the picture given in Figure 2.

We can see that a 45 degree angle corresponds to \( \theta = \pi/4 \) which gives \( x = 3 \) and a hypotheneuse length of \( r = 3\sqrt{2} \). We have the following given
Figure 2: You observe a boat from the shore as it travels north to south along a path 3 kilometers from the straight shoreline.

Information:

\[
\frac{dx}{dt} = -30
\]

\[
\frac{d\theta}{dt} = ?
\]

\[
\theta = \frac{\pi}{4}
\]

(at \( \theta = \pi/4 \)) \( x = 3 \)

(at \( \theta = \pi/4 \)) \( r = 3\sqrt{2} \).

In order to relate \( x \) to \( \theta \) we notice that

\[
\tan(\theta) = \frac{x}{3} \quad \Rightarrow \quad 3\tan(\theta) = x.
\]

If we differentiate this with respect to the independent variable \( t \) we have

\[
3\sec^2(\theta) \frac{d\theta}{dt} = \frac{dx}{dt}
\]
We can evaluate this at the given point to get

\[ 3 \sec^2 \left( \frac{\pi}{4} \right) \frac{d\theta}{dt} = -30 \]

\[ \Rightarrow (\sqrt{2})^2 \frac{d\theta}{dt} = -10 \Rightarrow \frac{d\theta}{dt} = -5. \]

It follows that we are pivoting at a rate of \(-5\) radians/hour at the point where the boat is making a 45 degree angle with the shoreline.