1 Curve Sketching

Consider the question of translating the algebraic equation for a function \( y = f(x) \) into a graph. There are a few standard functional forms for which we know the shape of the graph—polynomials, roots, exponentials, trigonometric functions, etc. But how do we handle functions which are combinations or these, or just too complicated to guess the shape of by looking at?

A first naive approach would be to build a table of \( x \) and \( y \) values, plot them, and then connect the dots. This a what a computer software package would do. This would be computational exhaustive to do by hand, however, and we might always wonder whether we had chosen a representative sample of points.

We are going to take a different approach. So far in this course, we have introduced a number of different ways to analyze functions. We have also paid attention to the graphical interpretation of these analyses. We know methods, for instance, which allow us to determine:

1. The domain and range of a function;

2. Horizontal and vertical asymptotes;

3. Intervals of positivity and negativity and roots (i.e. considerations of the sign of \( f(x) \));

4. Intervals of increase and decrease and critical points (i.e. consideration of the sign of \( f'(x) \)); and

5. Intervals of concavity and points of inflection (i.e. considerations of the sign of \( f''(x) \)).

In other words, the techniques we have developed thus far should give us plenty of information for sketching the curve of a function!
**Example 1:** Sketch the graph of \( f(x) = x^3 - 6x^2 + 9x - 4 \). Label important points.

**Solution:** We start by considering the asymptotes. There are clearly no vertical asymptotes, since there are no restrictions to the domain, and we have

\[
\lim_{x \to -\infty} f(x) = -\infty \quad \text{and} \quad \lim_{x \to \infty} f(x) = \infty.
\]

It follows that there are no horizontal asymptotes and that the function approaches starts at negative infinity and goes toward positive infinity as \( x \) goes from left to right. It follows that the domain and range are both the entire real numbers.

To find the roots, we notice that \( x^3 - 6x^2 + 9x - 4 \) can be factored. The most obvious choice of factor is \( x = 1 \), or the bracket \((x - 1)\), since the coefficients of the terms add to zero. By synthetic division, we have

\[
\begin{array}{c|cccc}
1 & 1 & -6 & 9 & -4 \\
 & & 1 & -5 & 4 \\
\hline
 & 1 & -5 & 4 & 0
\end{array}
\]

Since this works out exactly, we have \( f(x) = x^3 - 6x^2 + 9x - 4 = (x - 1)(x^2 - 5x + 4) \). We can easily factor the remaining quadratic to get \( f(x) = (x - 1)(x - 4)(x - 1) = (x - 1)^2(x - 4) \). It follows that the roots of the function are

**Roots:** \( \{1, 0\} \quad \{4, 0\} \)

Now consider the derivative of \( f(x) \). We have \( f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x - 3)(x - 1) \). We can plug the values \( x = 1 \) and \( x = 3 \) into \( f(x) \) to get \( f(1) = 0 \) and \( f(3) = -4 \). It follows that the critical values are

**Critical values:** \( \{(1, 0)\} \quad \{(3, -4)\} \)

Now consider the second derivative of \( f(x) \). We have \( f''(x) = 6x - 12 = 6(x - 2) \). It follows that the only point of inflection occurs at \( x = 2 \). We have \( f(2) = -2 \). It follows that

**Points of Inflection:** \( (2, -2) \).

We still need to consider the intervals of increase and decrease, and the intervals of concavity. Since \( f(x) \) and all of its derivatives are continuous,
Table 1: Intervals of increasing/decreasing and concavity

<table>
<thead>
<tr>
<th></th>
<th>$x &lt; 1$</th>
<th>$1 &lt; x &lt; 2$</th>
<th>$2 &lt; x &lt; 3$</th>
<th>$x &gt; 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f'(x)$</td>
<td>+</td>
<td>−</td>
<td>−</td>
<td>+</td>
</tr>
<tr>
<td>$f''(x)$</td>
<td>−</td>
<td>−</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>Shape</td>
<td>⨯</td>
<td>⨯</td>
<td>⋃</td>
<td>⋃</td>
</tr>
</tbody>
</table>

the only places the signs can change are at critical values and points of inflection. We will consider this as a single table. See Table 1.

We can now easily construct the graph of the function based on this information (see Figure 1).

Example 2: Sketch the graph of $f(x)$ based on the following graph of $f'(x)$. Capture the main qualitative features, including critical points and points of inflection. (See Figure 2.)
Solution: The sign of the derivative of $f(x)$ corresponds to intervals where the function is increasing and decreasing. Everywhere $f'(x)$ is above the axis corresponds to an interval of increase ($0 < x < 1$ and $1 < x < 3$), everything below the axis corresponds to an interval of decrease ($3 < x < 5$), while the critical points are the roots of the function ($x = 1$, $x = 3$, and $x = 5$). We also have that the critical points of $f'(x)$ correspond to points of inflection ($x = 2$ and $x = 4$).

We are free to choose any starting point for the function $f(x)$ we like, but we must follow the following general pattern:

1. $0 < x < 1$: $f(x)$ increases
2. $x = 1$: $f(x)$ levels off
3. $1 < x < 3$: $f(x)$ increases and has a point of inflection at $x = 2$
4. $x = 3$: $f(x)$ levels off
5. $3 < x < 5$: $f(x)$ decreases and has a point of inflection at $x = 4$
6. $x = 5$: $f(x)$ levels off.

We might arrive at the graph given in Figure 3.
Figure 3: A potential graph of a function which has the derivative given in Figure 2.