1 Anti-differentiation

Suppose we know the velocity of a car and wish to discover the position or the vehicle over time (i.e. how far the car has travelled). As a simple example, let’s suppose we know that the vehicle is travelling at a constant velocity of 80 km/h. We know that the velocity corresponds to the change in position over time, that is to say, we have

\[
\frac{dp(t)}{dt} = v(t) = 80.
\]

In order to determine for the position of the car, \(p(t)\), we need to find a function whose derivative is the constant value 80. In this case, we can probably just guess. We can easily see that the function \(p(t) = 80t\) works since \(p'(t) = 80\).

There is, however, one further subtlety. The function \(p(t) = 80t + 5\) works just as well, since the derivative of 5 is zero. In fact, any function of the form \(p(t) = 80t + C\) satisfies \(p'(t) = 80\). If we assume that our initial position (i.e. the position at time zero, \(p(0)\)) is zero, we can solve for \(C\) as \(p(0) = 0 = 80(0) + C \Rightarrow C = 0\) so that we have \(p(t) = 80t\). Without this addition information, however, we must leave the solution as \(p(t) = 80t + C\).

The process we have just outlined is called anti-differentiation and the function \(p(t)\) is called the anti-derivative of \(v(t)\). A general property is that we will have to add a general constant \(C\) to the anti-derivative. Sometimes further information will allow us to solve for this constant.

Anti-differentiation is one of the foundational concepts of calculus and is, in many ways, just as important as or more important than the process of differentiation. The applications extend far beyond simply finding functions which give a desired function as its derivative. It is one (fundamental!) part of a mathematical process called integration, which has a very specific notation.

**Definition 1.1.** Suppose that \(f(x)\) is such that \(F'(x) = f(x)\) for some function \(F(x)\). Then the indefinite integral of \(f(x)\) is defined to be

\[
\int f(x) \, dx = F(x) + C
\]
where $C$ is an arbitrary constant.

In other words, the indefinite integral is the set of all anti-derivatives of a function $f(x)$. There are a few notes worth making about this process:

1. Several standard properties which held for derivatives also apply for integration. In particular,

$$\int cf(x) \, dx = c \int f(x) \, dx$$

for constant values $c \in \mathbb{R}$, and

$$\int (f(x) \pm g(x)) \, dx = \int f(x) \, dx \pm \int g(x) \, dx.$$

2. In general, the process of integration is much more difficult than the process of differentiation. While there exist well-defined rules which can give us the derivative of nearly any function (e.g. product rule, chain rule, etc.), no such general rules exist for integrals. Many techniques exist for evaluating integrals, but not every function can be attained as the derivative of an easy-to-define function, even in principle.

3. Nevertheless, there is a wide class of functions for which we know the integral—every function we attained as the derivative of some basic form! We have

(a) \[ \frac{d}{dx} x^{n+1} = x^n, \quad \implies \quad \int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1 \]

(b) \[ \frac{d}{dx} \ln(x) = \frac{1}{x}, \quad \implies \quad \int \frac{1}{x} \, dx = \ln(x) + C \]

(c) \[ \frac{d}{dx} e^x = e^x, \quad \implies \quad \int e^x \, dx = e^x + C \]

(d) \[ \frac{d}{dx} \sin(x) = \cos(x), \quad \implies \quad \int \cos(x) \, dx = \sin(x) + C \]

(e) \[ \frac{d}{dx} \cos(x) = -\sin(x), \quad \implies \quad \int \sin(x) \, dx = -\cos(x) + C \]

(f) \[ \frac{d}{dx} \tan(x) = \sec^2(x), \quad \implies \quad \int \sec^2(x) \, dx = \tan(x) + C \]

(g) \[ \frac{d}{dx} \sec(x) = \tan(x) \sec(x), \quad \implies \quad \int \tan(x) \sec(x) \, dx = \sec(x) + C \]
4. When functions being integrating are not in any of these forms, our first step is to guess. We will learn better methods in one of the final lectures.

5. There is also the **definite integral**... details to come.

**Example 1:** Find the anti-derivative of \( \sin(7x) \).

**Solution:** We know that \( \sin(x) \) and \( \cos(x) \) turn into one another (with an occasional sign change) so we might try \( \cos(7x) \). We have

\[
\frac{d}{dx} \cos(7x) = -7 \sin(7x).
\]

We are close but not quite there. We divide both sides by \(-7\) to get

\[
\frac{d}{dx} \left[ -\frac{\cos(7x)}{7} \right] = \sin(7x).
\]

Adding our necessary constant, we have the final form \(-\frac{\cos(7x)}{7} + C\).

**Example 2:** Find the anti-derivative of \( 2 \sin(x) \cos(x) \).

**Solution:** We know the derivatives of \( \sin(x) \) and \( \cos(x) \) relate to one another, but it is difficult to see what we have to differentiate in order to get their product. We cannot, for instance, use the fact that

\[
\frac{d}{dx} \sin(x) = \cos(x), \quad \text{and} \quad \frac{d}{dx} \cos(x) = -\sin(x)
\]

to conclude that the antiderivative of \( 2 \sin(x) \cos(x) \) is \(-2 \cos(x) \sin(x)\) — this doesn’t work (Check!). In other words, we cannot just find the anti-derivate of each individual term.

In this case, we have to recognize that the candidate function may have arrived at the particular function in question via one of the product, quotient, or chain rules. The chain rule is the correct rule here, since we have that

\[
\frac{d}{dx} [\sin^2(x)] = 2 \sin(x) \cos(x).
\]
The correct antiderivative is therefore \( \sin^2(x) + C \).

**Bonus:** There is another answer to this question. We might also notice that
\[
\frac{d}{dx}[-\cos^2(x)] = 2\sin(x)\cos(x).
\]
So the anti-derivative \(-\cos^2(x) + C\) also works!

How is this possible? How can we have two different antiderivatives to the same problem (we know that \( \sin^2(x) \neq \cos^2(x) \))? The answer lies in the fact that we have no specified the values of the constants. We know that
\[
\sin^2(x) + \cos^2(x) = 1
\]
so that \( \sin^2(x) = -\cos^2(x) + 1 \). If we add an arbitrary constant \( C \) to the left-hand and right-hand sides of this, we have
\[
\sin^2(x) + C = -\cos^2(x) + (C + 1).
\]
In other words, the solutions *are* equal, but we need to make the constant for the \(-\cos(x)\) term one higher than the constant for the \(\sin(x)\) term.

**Example 3:** Evaluate the indefinite integral
\[
\int \frac{1}{\sqrt{9-x^2}} \, dx.
\]

**Solution:** This is very close to the form required of \( \arcsin(x) \), since
\[
\frac{d}{dx} \arcsin(x) = \frac{1}{\sqrt{1-x^2}}.
\]
It is only the constant 9 which is throwing things off. We can rearrange our integral into a closer form by putting out the 9 to get
\[
\int \frac{1}{\sqrt{9-x^2}} \, dx = \int \frac{1}{3\sqrt{1-(\frac{x}{3})^2}} \, dx = \frac{1}{3} \int \frac{1}{\sqrt{1-(\frac{x}{3})^2}} \, dx.
\]
We might guess that this is the antiderivative of \( \arcsin(x/3) \), or something closely related to it. We check to get
\[
\frac{d}{dx} \arcsin \left( \frac{x}{3} \right) = \frac{1}{\sqrt{1-(\frac{x}{3})^2}} \cdot \frac{1}{3} = \frac{1}{3\sqrt{1-(\frac{x}{3})^2}}.
\]
It follows that
\[
\int \frac{1}{\sqrt{9-x^2}} \, dx = \arcsin \left( \frac{x}{3} \right) + C.
\]