1 Integration By Substitution

One principle technique of integration is integration by substitution. The central idea is that, while a function may not be readily integrable with respect to its current variable, there may be another variable we can integrate with respect to for which the integration step is much easier. Integration by substitution is justified by the chain rule — we won’t cover this in detail.

The algorithm for integration by substitution is:

1. Choose your new variable (say, \( u = g(x) \))

2. Replace every instance of the old variable with the new (including \( dx \), etc.)

3. Integrate with respect to the new variable

4. Replace the new variable with the old

5. Check the answer! (Take the derivative.)

There are a few things to note here:

1. The new variable is only needed for the integration step. The final answer should be stated with respect to the old variable. *Remember to switch back!*

2. Remember that the quantity to be integrated must be stated entirely in terms of the new variable. *Every* instance of the old variable must be replaced with the new. Under no circumstances may you integrate with respect to mixed variables. If you are unable to entirely replace the old variable with the new, you have probably chosen a poor change of variable and should try something else.

3. Substitution (and integration in general) is often a matter of trial and error. A general principle is that the most troublesome term in the expression (e.g. something under a root or in a denominator) is a good candidate for substitution; however, this does not always hold. If your first substitution does not work, try another one!
Example 1:

Consider the integral
\[ \int x \sqrt{2x - 1} \, dx. \]

If you can immediately recognize a function whose derivative is \( x \sqrt{2x - 1} \), you are a better mathematician than I am. Instead, we should try a substitution, but what should we pick? The thing to note here is that our real trouble point is the square root, since it does not distribute across \( 2x - 1 \). The integral would be easier if the root were with respect to a single variable, so let’s choose

\[ u = 2x - 1. \]

Remember, however, that we need to replace all instances of \( x \) from original integral. Since \( u \) depends on \( x \), we can evaluate

\[ \frac{du}{dx} = 2 \implies dx = \frac{1}{2} du. \]

All that remains is to substitute is the \( x \), which can be found by rearranging the original substitution to give

\[ x = \frac{1}{2}(u + 1). \]

We have everything we need, so let’s plug it into the original integral:

\[ \int \frac{1}{2} (u + 1) u^{1/2} \frac{1}{2} du = \frac{1}{4} \int \left[u^{3/2} + u^{1/2}\right] du. \]

Everything under the integral is in terms of the new variable \( u \) so we can integrate over \( u \) to get

\[ \frac{1}{4} \left[ \frac{2}{5} u^{5/2} + \frac{2}{3} u^{3/2} \right] + C = \frac{1}{10} u^{5/2} + \frac{1}{6} u^{3/2} + C. \]

That was surprisingly painless, but we are not quite done. The original expression was in terms of \( x \), so that is how we should state the answer. We only needed the new variable to simplify the integration step. Using the original substitution, our final answer is

\[ \frac{1}{10} (2x - 1)^{5/2} + \frac{1}{6} (2x - 1)^{3/2} + C. \]
But how do we know this is correct? We don’t, after all, necessarily have a pre-set idea what the integral should look like. We can, however, simply take the derivative and hope we arrive at the original expression. Since taking derivatives is easy to do, it usually does not take very long to check our answer. This should be a common practice!

For this example, we have

\[
\frac{d}{dx} \left[ \frac{1}{10} (2x - 1)^{5/2} + \frac{1}{6} (2x - 1)^{3/2} + C \right] \\
= 2 \left( \frac{1}{10} \right) \left( \frac{5}{2} \right) (2x - 1)^{3/2} + 2 \left( \frac{1}{6} \right) \left( \frac{3}{2} \right) (2x - 1)^{1/2} \\
= \frac{1}{2} (2x - 1)^{3/2} + \frac{1}{2} (2x - 1)^{1/2} \\
= \frac{1}{2} (2x - 1)^{1/2} (2x - 1 + 1) \\
= x(2x - 1)^{1/2}.
\]

So our answer is correct.

**Example 2:**

Integration by substitution can also be applied to definite integrals; however, we need to keep track of how the bounds of integration change as a result of the substitution. After our change of variable, we have two options in this regard:

1. Find the new bounds, integrate, and solve using the new bounds directly.
2. Ignore the bounds, integrate, replace new variable with old and solve using the old bounds.

Which technique you use is up to you, but you MUST be clear what the bounds are. If you mix up the bounds for the variables, you will likely end up with a spectacularly wrong answer.

Consider the definite integral

\[
\int_{0}^{9} \frac{(1 + \sqrt{x})^{1/2}}{\sqrt{x}} \, dx.
\]

Again, the problem with integrating this directly is the root, so we try the substitution

\[
u = 1 + \sqrt{x}.
\]
This readily leads to
\[ \frac{du}{dx} = \frac{1}{2\sqrt{x}} \implies dx = 2\sqrt{x}du. \]

However, we still need to consider the bounds. When \( x = 0 \) we have \( u = 1 \) and when \( x = 9 \) we have \( u = 4 \), so when we substitute \( u \) for \( x \), we are now integrating from 1 to 4.

We have
\[
2 \int_{1}^{4} u^{1/2} \, du
= \left[ \frac{4}{3} u^{3/2} \right]_{1}^{4}
= \frac{4}{3} \left( 4^{3/2} - 1^{3/2} \right) = \frac{28}{3}.
\]

We could have neglected the change in the bounds but we would have had to have remembered to change back to the original variable after integrating before evaluating at the bounds. For example,
\[
\int_{0}^{9} \frac{(1 + \sqrt{x})^{1/2}}{\sqrt{x}} \, dx
= 2 \int_{x=0}^{x=9} u^{1/2} \, du
= \left[ \frac{4}{3} u^{3/2} \right]_{x=0}^{x=9}
= \left[ \frac{4}{3} (1 + \sqrt{x})^{3/2} \right]_{0}^{9} = \frac{28}{3}.
\]