Math 118 - Lab 4 - Winter 2009.
Improper Integrals and parametric equations.

You are to provide full solutions to the following problems. You are allowed to collaborate with your classmates, use your notes and textbook and ask the TA for guidance. Direct copying of solutions is not encouraged, nor is it allowed or ethical.

Last name: ___________________________ First name: ___________________________

Student number: _______________________

(Please indicate your student number on the first page of the solutions, but not your name.)
1. Determine if the improper integral $\int_{0}^{\pi/2} \tan(x) \, dx$ converges or diverges; if it converges, determine its value.

**Solution:** The only point in the interval $[0, \pi/2]$ at which $\tan(x)$ is undefined is $x = \pi/2$. Thus, the relevant limit is

$$\lim_{z \to (\pi/2)^-} \int_{0}^{z} \tan(x) \, dx.$$  

As $\int \tan(x) \, dx = -\ln |\cos(x)| + C$, we have that the relevant limit is

$$\lim_{z \to (\pi/2)^-} (-\ln |\cos(z)| + \ln |\cos(0)|).$$

Since $\ln |\cos(0)| = 0$ and $\lim_{z \to (\pi/2)^-} \ln |\cos(z)|$ diverges to $-\infty$, $\int_{0}^{\pi/2} \tan(x) \, dx$ diverges to $+\infty$.

2. Determine if the improper integral

$$\int_{2}^{\infty} \frac{1}{x \ln(x)} \, dx$$

converges or diverges; if it converges, determine its value.

**Solution:** Let

$$I = \lim_{z \to \infty} \int_{2}^{z} \frac{1}{x \ln(x)} \, dx$$

and note that the substitution $u = \ln(x)$ converts

$$\int_{2}^{z} \frac{1}{x \ln(x)} \, dx$$

into

$$\int_{\ln(2)}^{\ln(z)} \frac{1}{u} \, du = \ln (\ln(z)) - \ln (\ln(2)).$$

Since $\lim_{z \to \infty} \ln(\ln(z))$ diverges to $\infty$, $I$ diverges.
3. Determine if the improper integral

\[ \int_{-1}^{1} \frac{1}{x^2} \, dx \]

converges or diverges; if it converges, determine its value.

**Solution:** The problem is that \( 1/x^2 \) is undefined at \( x = 0 \). The integral converges precisely if both the limits

\[ \lim_{c \to 0^-} \int_{-1}^{c} \frac{1}{x^2} \, dx \quad \text{and} \quad \lim_{d \to 0^+} \int_{d}^{1} \frac{1}{x^2} \, dx \]

exist.

As

\[ \int_{a}^{b} \frac{1}{x^2} \, dx = \frac{1}{a} - \frac{1}{b}, \]

the first limit is \(-1 - \lim_{c \to 0^-} 1/c\), which is undefined (or diverges to \(+\infty\)). Thus, the integral diverges.

4. Use the fact that

\[ 0 \leq \frac{1}{x^2 \ln(x)} \leq \frac{1}{x^2} \]

for \( 3 \leq x \leq \infty \) to show that

\[ \int_{3}^{\infty} \frac{1}{x^2 \ln(x)} \, dx \]

converges, without actually giving its value.

**Solution:** The inequality implies

\[ \int_{3}^{\infty} \frac{1}{x^2 \ln(x)} \, dx \leq \int_{3}^{\infty} \frac{1}{x^2} \, dx \]

assuming the right-hand improper integral exists. We can evaluate it as follows:

\[ \int_{3}^{\infty} \frac{1}{x^2} \, dx = \lim_{z \to \infty} \int_{3}^{z} \frac{1}{x^2} \, dx = \lim_{z \to \infty} \left( \frac{1}{3} - \frac{1}{z} \right) = \frac{1}{3}. \]

Therefore

\[ \int_{3}^{\infty} \frac{1}{x^2 \ln(x)} \, dx \]

converges.
5. Draw the curve \( x = 1 + 2 \cos(t), \ y = 2 + 3 \sin(t) \), for \( 0 \leq t \leq 2\pi \).

**Solution:** In this case, we can rewrite so that \( \cos(t) = (x-1)/2 \) and \( \sin(t) = (y-2)/3 \), and, therefore,

\[
\left( \frac{x - 1}{2} \right)^2 + \left( \frac{y - 2}{3} \right)^2 = 1,
\]

which is the equation of an ellipse. Alternatively, we can plot points to determine the general shape, as given roughly in the figure.

6. Draw the curve \( x = t - t^2, \ y = t + t^2, \ t \geq 0 \).

**Solution:** We plot some points:

\[
\begin{array}{c|c|c|c|c|c|c}
 t & x(t) & y(t) \\
 \hline
 0 & 0 & 0 \\
 1 & 1 & 3 \\
 2 & 2 & 2 \\
 3 & 6 & 12 \\
 4 & 12 & 20 \\
\end{array}
\]
The curve looks something like the illustration.

Figure 2: The curve $x = t - t^2$, $y = t + t^2$

7. Find $dy/dx$ for the curve $x = t^{3/2} - t^{2/3}$, $y = t^2 + 2t$, $t \geq 0$.

**Solution:** We have that $dy/dt = 2t + 2$ and $dx/dt = (3/2)t^{1/2} - (2/3)t^{-1/3}$. Therefore,

$$
\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \frac{2t + 2}{(3/2)t^{1/2} - (2/3)t^{-1/3}} = \frac{6t^{1/3}(2t + 2)}{9t^{5/6} - 4}.
$$

8. Find all the points on the curve $x = e^t \cos(t)$, $y = e^t \sin(t)$ at which the slope of the tangent line is equal to 1.

**Solution:** We see that $dx/dt = e^t \cos(t) - e^t \sin(t)$ and $dy/dt = e^t \sin(t) + e^t \cos(t)$. Since $dy/dx$ is $(dy/dt)/(dx/dt)$, we see that $dy/dx = 1$ precisely when $dy/dt = dx/dt$. Since $e^t \neq 0$, this occurs precisely when $\cos(t) - \sin(t) = \sin(t) + \cos(t)$ and, therefore, $\sin(t) = 0$. This happens whenever $t = n\pi$, for $n = \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$.

For these values of $t$, we have $x = e^{n\pi}$ and $y = 0$. 

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9. Two particles are each moving along a curve, the first is on \( x = -t + t^2 + 1, \ y = -t \) and the second on \( x = t^2 - t^3, \ y = -t^3 - 1 \). For \( 0 \leq t \leq 1 \), at what time \( t \) are they closest?

**Solution:** The distance at a given time between the points is

\[
d(t) = \sqrt{(-t + t^2 + 1 - (t^2 - t^3))^2 + (-t - (-t^3 - 1))^2}
\]

\[
= \sqrt{(-t + t^3 + 1)^2 + (-t + t^3 + 1)^2}
\]

\[
= \sqrt{2((-t + t^3 + 1))}.
\]

For \( 0 \leq t \leq 1 \), \( 0 \leq t^3 \leq t \leq 1 \), so \( d(t) = \sqrt{2}(1 + t^3 - t) \).

We compute \( d'(t) \). We note that \( d'(t) = 3t^2 - 1 \), so \( d'(t) = 0 \) when \( t = \sqrt{1/3} \). For \( t \) strictly between 0 and 1, \( d''(t) > 0 \), so \( d \) is concave up, showing the minimum occurs at \( t = \sqrt{1/3} \), so the answer is the shortest distance is

\[
d(\sqrt{1/3}) = \sqrt{2} \left(1 - \frac{2}{3\sqrt{3}}\right).
\]