
Topics:

• Sums of power series

• Applications of Taylor series

You are to provide full solutions to the following problems. You are allowed, and encouraged to collaborate with your classmates, use your notes and textbook and ask the TA for guidance. Direct copying of solutions is not encouraged, nor is it allowed or ethical.

Last name: __________________________  First name: __________________________

Student number: __________________________
1. Use Taylor’s Remainder Theorem to estimate the maximum error of the third-order (two-term) Maclaurin series expansion of \( \sin(x/2) \) on the interval \( 0 < x < \pi/2 \).

2. Determine how many terms in the Taylor expansion are required to calculate \( \int_0^1 \sqrt{xe^x} \, dx \) accurately to within a bound of \( 10^{-6} \).

3. Use Taylor series to determine the following limits:
   
   (a) \( \lim_{x \to 0} \frac{x^2}{e^x - x - 1} \)  
   (b) \( \lim_{x \to 0} \frac{\sin(x)}{\ln(1 - x)} \)

4. Verify that \( \sum_{n=0}^{\infty} (-1)^n \frac{3^n}{2^{2n-1}} = \frac{8}{7} \).

5. Verify that \( \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \pi^{2n} = -1 \).

6. Find the Maclaurin series for \( \frac{x^2}{(1 - x^3)^2} \) and use it to evaluate \( \sum_{n=1}^{\infty} 2n \left( \frac{1}{2} \right)^{3n} \).

7. In this question, we are going to use power series to find a solution \( x(t) \) of the differential equation
   
   \( \frac{dx}{dt} = kx, \quad x(0) = 1 \). \hspace{1cm} (1)

   We begin by assuming the solution has a power series representation
   
   \( x(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots \) \hspace{1cm} (2)

   (a) Use the initial condition \( x(0) = 1 \) to solve for \( a_0 \).

   (b) Using (2), determine the power series representation of \( x'(t) \).

   (c) Plug these two series into (1). Solve for the terms \( a_n \) by equating coefficients of the powers of \( t \).

   (d) Rewrite the original power series \( x(t) \) with these new coefficients. To which function \( x(t) \) does this series correspond? Show that this is a solution by verifying it satisfies the two conditions given in (1).

   (e) **Bonus:** Use the same technique to find the solution of

   \( \frac{d^2x}{dt^2} + k^2 x = 0, \quad x(0) = 1, \ x'(0) = 0. \)