1. Use Taylor’s Remainder Theorem to estimate the maximum error of the third-order (two-term) Maclaurin series expansion of \( \sin(x/2) \) on the interval \( 0 < x < \pi/2 \).

Solution: The third-order Maclaurin series expansion of \( \sin(x/2) \) is

\[
P_3(x) = \sum_{n=0}^{1} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \frac{x}{2} - \frac{x^3}{48}
\]

The error is given by

\[
R_3(x) = \frac{f^{(4)}(z_n)}{4!} x^4 = \frac{\sin(z_n)}{2^4 4!} x^4
\]

where \( 0 < z_n < x < \pi/2 \). This can be bound so that

\[
|R_3(x)| = \left| \frac{\cos(z_n)}{2^4 4!} x^4 \right| < \frac{\left(\frac{\pi}{2}\right)^4}{2^4 4!} = \frac{\pi^4}{6144} \approx 0.015854.
\]

This implies that

\[
\frac{x}{2} - \frac{x^3}{48} - 0.015854 < \sin \left(\frac{x}{2}\right) < \frac{x}{2} - \frac{x^3}{48} + 0.015854
\]

on the interval \( 0 < x < \pi/2 \).

You could also use

\[
R_4(x) = \frac{f^{(5)}(z_n)}{5!} x^5 = \frac{\cos(z_n)}{2^5 5!} x^5
\]

which gives the bound

\[
|R_4(x)| < \frac{\pi^5}{122880} \approx 0.00249.
\]

2. Determine how many terms in the Taylor expansion are required to calculate \( \int_{0}^{1} \sqrt{xe^x} \, dx \) accurately to within a bound of \( 10^{-6} \).

Solution: We know that

\[
\int_{0}^{1} \sqrt{xe^x} \, dx = \int_{0}^{1} \sqrt{x} P_n(x) \, dx + \int_{0}^{1} \sqrt{x} R_n(x) \, dx.
\]
where $P_n(x)$ is the $n^{th}$ order Taylor polynomial for $e^x$ and $R_n(x)$ is the $n^{th}$ order Taylor remainder. We need to bound the remainder so that

$$\left| \int_0^1 \sqrt{x} R_n(x) \, dx \right| < 10^{-6}.$$  

We have

$$\left| \int_0^1 \sqrt{x} R_n(x) \, dx \right| = \left| \int_0^1 \sqrt{x} \frac{f^{(n+1)}(z_n)}{(n+1)!} x^{n+1} \, dx \right|$$

$$= \left| \int_0^1 e^{z_n} x^{n+3/2} \, dx \right|$$

$$< \left| \int_0^1 e^{x^{n+3/2}} \, dx \right|$$

$$= \left[ \frac{e^{x^{n+5/2}}}{(n+1)! (n+5/2)} \right]_0^1$$

$$= \frac{e}{(n+1)! (n+5/2)} < 10^{-6}.$$  

This cannot be solved directly, but we can easily evaluate a few terms with our calculator to find that the first such value satisfying the desired relation is $n = 8$.

3. Use Taylor series to determine the following limits:

(a) $\lim_{x \to 0} \frac{x^2}{e^x - x - 1}$  
(b) $\lim_{x \to 0} \frac{\sin(x)}{\ln(1-x)}$

**Solution (a):** We substitute the Maclaurin series $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ to get

$$\lim_{x \to 0} \frac{x^2}{e^x - x - 1} = \lim_{x \to 0} \frac{x^2}{(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots) - x - 1}$$

$$= \lim_{x \to 0} \frac{x^2}{\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots}$$

$$= \lim_{x \to 0} \frac{1}{\frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \cdots} = 2.$$  

**Solution (b):** We substitute the Maclaurin series $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$ and $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$ to get

$$\lim_{x \to 0} \frac{\sin(x)}{\ln(1-x)} = \lim_{x \to 0} \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}{-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots}$$

$$= \lim_{x \to 0} \frac{1 - \frac{x^2}{2} + \frac{x^4}{4} - \cdots}{-\frac{x}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots} = -1.$$
4. Verify that $\sum_{n=0}^{\infty} (-1)^n \frac{3^n}{2^{2n-1}} = \frac{8}{7}$.

**Solution:** We rearrange the series to get

$$\sum_{n=0}^{\infty} (-1)^n \frac{3^n}{2^{2n-1}} = 2 \sum_{n=0}^{\infty} \frac{(-3)^n}{(2)^n} = \sum_{n=0}^{\infty} \left( -\frac{3}{4} \right)^n.$$

We recognize this as the geometric series

$$\sum_{n=0}^{\infty} ax^n$$

with $x = -3/4$. The series converges for $|x| < 1$ which is satisfied for $x = -3/4$, so the series converges. We have

$$\sum_{n=0}^{\infty} ax^n = \frac{a}{1-x} \implies \sum_{n=0}^{\infty} 2 \left( -\frac{3}{4} \right)^n = \frac{2}{1+3/4} = \frac{8}{7}.$$

5. Verify that $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \pi^{2n} = -1$.

**Solution:** The Maclaurin series expansion of $\cos(x)$ is given by

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

so that, setting $x = \pi$, we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \pi^{2n} = \cos(\pi) = -1.$$

6. Find the Maclaurin series for $x^2/(1-x^3)^2$ and use it to evaluate $\sum_{n=1}^{\infty} 2n \left( \frac{1}{2} \right)^{3n}$.

**Solution:** We do not know the Maclaurin series for anything of the form $x^2/(1-x^3)^2$; however, we do know how to integrate this. We have

$$f(x) = \frac{x^2}{(1-x^3)^2}$$

$$\implies \int f(x) \, dx = \frac{1}{3(1-x^3)} + C = \frac{1}{3} \sum_{n=0}^{\infty} x^{3n} + C = \frac{1}{3} + \frac{1}{3} \sum_{n=1}^{\infty} x^{3n} + C.$$
\[ f(x) = \frac{d}{dx} \int f(x) \, dx = \sum_{n=1}^{\infty} n x^{3n-1}. \]

This expansion converges for (at least) \(-1 < x < 1\) since this is where \(1/(3(1 - x^3))\) converged.

Consider this expansion for the value \(x = 1/2\). We have

\[ \sum_{n=1}^{\infty} n \left( \frac{1}{2} \right)^{3n-1} = \sum_{n=1}^{\infty} 2n \left( \frac{1}{2} \right)^{3n} \).

This is exactly the sum we are asked to evaluate, so we have

\[ \sum_{n=1}^{\infty} 2n \left( \frac{1}{2} \right)^{3n} = \frac{(1/2)^2}{(1 - (1/2)^3)^2} = \frac{1/4}{49/64} = \frac{16}{49}. \]

7. In this question, we are going to use power series to find a solution \(x(t)\) of the differential equation

\[ \frac{dx}{dt} = kx, \quad x(0) = 1. \] \hfill (1)

We begin by assuming the solution has a power series representation

\[ x(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots \] \hfill (2)

(a) Use the initial condition \(x(0) = 1\) to solve for \(a_0\).

(b) Using (2), determine the power series representation of \(x'(t)\).

(c) Plug these two series into (1). Solve for the terms \(a_n\) by equating coefficients of the powers of \(t\).

(d) Rewrite the original power series \(x(t)\) with these new coefficients. To which function \(x(t)\) does this series correspond? Show that this is a solution by verifying it satisfies the two conditions given in (1).

(e) **Bonus:** Use the same technique to find the solution of

\[ \frac{d^2x}{dt^2} + k^2x = 0, \quad x(0) = 1, \quad x'(0) = 0. \]

**Solution (a):** Since \(x(0) = 1\) we have

\[ 1 = x(0) = a_0 + a_1(0) + a_2(0)^2 + \cdots = a_0. \]

**Solution (b):** We assume that the solution can be represented in the form of a power series, that is to say

\[ x(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots \]
which implies that
\[ x'(t) = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + \cdots. \]

**Solution (c):** In order for this to satisfy our differential equations, this must satisfy
\[
(a_1 + 2a_2 t + 3a_3 t^2 + \cdots) - k(1 + a_1 t + a_2 t^2 + \cdots) = 0
\]
\[
\implies (a_1 - k) + (2a_2 - ka_1)t + (3a_3 - ka_2)t^2 + \cdots = 0.
\]
In order for this to be satisfied, we need all of the coefficients on the land-hand side to vanish, so that we have the system
\[
a_1 - k = 0 \implies a_1 = k
\]
\[
2a_2 - ka_1 = 0 \implies a_2 = \frac{k^2}{2}
\]
\[
3a_3 - ka_2 = 0 \implies a_3 = \frac{k^3}{3!}
\]
\[
\vdots
\]
\[
a_n - ka_{n-1} = 0 \implies a_n = \frac{k^n}{n!}.
\]

**Solution (d):** Our power series solution looks like
\[
x(t) = 1 + kx + \frac{(kx)^2}{2!} + \frac{(kx)^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{(kx)^n}{n!} = e^{kx}.
\]
This is a solution since \( x'(t) = ke^{kx} \) so that
\[
\frac{dx}{dt} = ke^{kx} = kx(t).
\]
We also have \( x(0) = e^{k(0)} = 1. \)

**Solution (e):** We assume the differential equation has a power series solution of the form
\[
x(t) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots.
\]
This implies that
\[
x'(t) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_4 x^4 + \cdots
\]
and
\[
x''(t) = 2a_2 + 2 \cdot 3a_3 x + 3 \cdot 4a_4 x^2 + 4 \cdot 5a_4 x^3 + \cdots.
\]
Clearly, plugging in \( x(0) = 1 \) implies \( a_0 = 1 \) and \( x'(0) = 0 \) implies \( a_1 = 0 \).

We plug these series into the differential equation to get

\[
(2a_2 + 2 \cdot 3a_3x + \cdots) + k^2(1 + a_2x^2 + a_3x^3 + \cdots) = 0
\]

\[
\implies (2a_2 + k^2) + (2 \cdot 3a_3)x + (3 \cdot 4a_4 + k^2a_2)x^2 +
\]

\[
(4 \cdot 5a_5 + k^2a_3)x^3 + (5 \cdot 6a_6 + k^2a_4)x^4 + \cdots = 0.
\]

The even coefficients only depend on the other even coefficients, while the odd coefficients only depend on the other odd coefficients. Furthermore, we can see that \( a_1 = 0 \) implies \( a_3 = 0 \), which implies \( a_5 = 0 \), and so on, so that we only have to worry about the even coefficients. For the even coefficients we have

\[
a_2 = -\frac{k^2}{2!} \implies a_4 = \frac{k^4}{4!} \implies \ldots \implies a_n = (-1)^{n/2} \frac{k^n}{n!}.
\]

Plugging these coefficients into the power series expansion for \( x(t) \) gives

\[
x(t) = 1 - \frac{(kx)^2}{2!} + \frac{(kx)^4}{4!} - \frac{(kx)^6}{6!} + \cdots = \cos(kx)
\]

which is a solution to the differential equation (and initial conditions).