1. Find the general solution of $\frac{dy}{dx} = \frac{1}{(1 - 4x^2)}$.

**Solution:** We have

$$\frac{dy}{dx} = \frac{1}{(1 - 4x^2)} = \frac{1}{(1 - 2x)(1 + 2x)} \implies \int 1 \, dy = \int \frac{1}{(1 - 2x)(1 + 2x)} \, dx.$$ 

To solve the right-hand side integral, we use the substitution $u = 1 + 2x$. This gives $du = 2 \, dx$ and $1 - 2x = u - 4x = u - 2(u - 1) = 2 - u$. We have

$$\int \frac{1}{(1 - 2x)(1 + 2x)} \, dx = \int \frac{1}{2u(2 - u)} \, du.$$ 

We can expand this using partial fractions to get

$$\frac{1}{u(2 - u)} = \frac{A}{u} + \frac{B}{2 - u} \implies 1 = A(2 - u) + Bu.$$ 

Setting $u = 0$ gives $A = 1/2$ and setting $u = 2$ gives $B = 1/2$. So we have

$$\int \frac{1}{2u(2 - u)} \, du = \int \left[ \frac{1}{4u} + \frac{1}{4(2 - u)} \right] \, du = \frac{1}{4} \ln(u) - \frac{1}{4} \ln(2 - u) + C = \frac{1}{4} \ln \left( \frac{u}{2 - u} \right) + C = \frac{1}{4} \ln \left( \frac{1 + 2x}{1 - 2x} \right) + C.$$ 

Consequently, the general solution is

$$y(x) = \frac{1}{4} \ln \left( \frac{1 + 2x}{1 - 2x} \right) + C.$$ 

2. Solve the initial value problem $\frac{dy}{dx} = \frac{1 - y^3}{xy^2}$, $y(0) = 0$. 


Solution: We have
\[ \frac{dy}{dx} = \frac{1 - y^3}{xy^2} \]
\[ \Rightarrow \int \frac{y^2}{1 - y^3} \, dy = \int \frac{1}{x} \, dx \]
\[ \Rightarrow -\frac{1}{3} \ln(1 - y^3) = \ln(x) + C. \]
The initial condition \( y(1) = 0 \) implies \(-1/3) \ln(1) = \ln(1) + C \) which implies \( C = 0 \).
We have
\[ -\frac{1}{3} \ln(1 - y^3) = \ln(x) \]
\[ \Rightarrow \ln \left( \frac{1}{\sqrt[3]{1 - y^3}} \right) = \ln(x) \]
\[ \Rightarrow \sqrt[3]{1 - y^3} = \frac{1}{x} \]
\[ \Rightarrow 1 - y^3 = \frac{1}{x^3} \]
\[ \Rightarrow y(x) = 3 \sqrt[3]{x^3 - 1} = \frac{1}{x} \sqrt[3]{x^3 - 1}. \]

3. Show that \( y(x) = e^{-x}(\cos(x) + \sin(x)) \) is a solution of \( \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + 2y = 0. \)

Solution: We need the following terms
\[
y(x) = e^{-x}(\cos(x) + \sin(x))
\]
\[
y'(x) = -e^{-x}(\cos(x) + \sin(x)) + e^{-x}(-\sin(x) + \cos(x))
\]
\[ = -2e^{-x} \sin(x) \]
\[
y''(x) = 2e^{-x}(\sin(x) - \cos(x)).
\]
Plugging into the differential equation, we have
\[
\frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + 2y
\]
\[ = 2e^{-x}(\sin(x) - \cos(x)) - 4e^{-x} \sin(x) + 2e^{-x}(\cos(x) + \sin(x)) \]
\[ = e^{-x}(2 \sin(x) - 4 \sin(x) + 2 \sin(x) - 2 \cos(x) + 2 \cos(x)) \]
\[ = 0. \]

4. Find the general solution of the differential equation \( \frac{dy}{dx} + \frac{y}{x} = \cos(x). \)
Solution: This is a linear first-order system, so we need to find an integrating factor. It is given by
\[ \mu(x) = e^{\int (1/x) \, dx} = e^{\ln(x)} = x. \]

We multiply this across the differential equation to get
\[
x \frac{dy}{dx} + y = x \cos(x) \\
\Rightarrow \frac{d}{dx} [xy] = x \cos(x) \\
\Rightarrow xy = \int x \cos(x) \, dx \\
\Rightarrow x \sin(x) - \int \sin(x) \, dx \\
\Rightarrow y(x) = \sin(x) + \frac{\cos(x)}{x} + C.
\]

5. Solve the initial value problem \( \frac{d^2 y}{dx^2} + \frac{dy}{dx} = e^{-x}, \ y(0) = -1, \ y'(0) = -1. \)

Solution: This equation has the dependent variable \( y \) missing, so we set \( v = y' \) (which implies \( v' = y'' \)). We have
\[ \frac{dv}{dx} + v = e^{-x}. \]

This is a linear first-order equation in \( v \), so we need to find the integrating factor. It is given by
\[ \mu(x) = e^{\int 1 \, dx} = e^x. \]

We multiply across the equation by \( e^x \) to get
\[
e^x \frac{dv}{dx} + e^x v = 1 \\
\Rightarrow \frac{d}{dx} [e^x v] = 1 \\
\Rightarrow e^x v = x + C \\
\Rightarrow \frac{dy}{dx} = v = xe^{-x} + Ce^{-x} \\
\Rightarrow y(x) = -xe^{-x} - e^{-x} - Ce^{-x} + D.
\]

The initial condition \( y'(0) = -1 \) can be applied to the \( \frac{dy}{dx} \) expression to give \(-1 = (0)e^0 + Ce^0 \Rightarrow C = -1. \) This simplifies the solution to
\[ y(x) = -xe^{-x} + D. \]
The condition \( y(0) = -1 \) implies 

\[ -1 = -(0)e^0 + D \implies D = 0 \]

so that we have

\[ y(x) = -xe^{-x}. \]

6. Solve the initial value problem

\[ \frac{d^2y}{dx^2} e^y - \frac{dy}{dx} = 0, \quad y(0) = 0, \quad y'(0) = -1. \]

(Hint: Use the initial conditions simultaneously to solve for the first constant of integration.)

**Solution:** This equation has the independent variable \( x \) missing, so we set \( v = y' \) (which implies \( y'' = vv' \), where the \( v \) derivative is with respect to \( y \)). We have

\[
\frac{d^2y}{dx^2} e^y - \frac{dy}{dx} = 0
\]

\[ \implies v \frac{dv}{dy} e^y = v
\]

\[ \implies \int 1 \, dv = \int e^{-y} \, dy
\]

\[ \implies \frac{dy}{dx} = v = -e^{-y} + C.
\]

Integrating this directly is very difficult with the constant \( C \) there—however, we can solve for \( C \) by noticing that at \( x = 0 \) both \( y(x) \) and \( y'(x) \) are solved for! The conditions \( y(0) = 0 \) and \( y'(0) = -1 \) give the equation

\[ -1 = -(0)e^0 + C \]

which implies \( C = 0 \). This simplifies our integration significantly. We now have

\[ \frac{dy}{dx} = -e^{-y}
\]

\[ \implies \int e^y \, dy = - \int 1 \, dx
\]

\[ \implies e^y = x + C.
\]

The condition \( y(0) = 0 \) implies \( e^0 = (0) + C \) so that \( C = 1 \). Our solution is therefore given by

\[ y(x) = \ln(1 - x).
\]

7. (a) Solve the system of differential equations

\[
\frac{dx}{dt} = -kx, \quad \frac{dy}{dt} = kx, \quad x(0) = 1, \quad y(0) = 0.
\]

(Hint: Since the first equation depends only on \( x \), solve it first.)

(b) Under suitable laboratory conditions, this models the time evolution of the reaction

\[ A \xrightarrow{k} B \]
where \( x = [A] \) and \( y = [B] \) are concentrations of the two chemical reactants and \( A \) is present initially while \( B \) is not. Plot the solutions and briefly explain what happens to the chemicals as time passes (take \( k = 1 \), if it helps).

**Solution (a):**  The first equation is separable in \( x \) and \( t \). We obtain

\[
\frac{dx}{x} = -k \, dt
\]

\[
\Rightarrow \quad \int \frac{dx}{x} = - \int k \, dt
\]

\[
\Rightarrow \quad \ln(x) = -kt + C
\]

\[
\Rightarrow \quad x(t) = e^C e^{-kt}.
\]

The initial condition \( x(0) = 1 \) implies \( e^C = 1 \) so that we have

\[
x(t) = e^{-kt}.
\]

We plug this into the second equation to get

\[
\frac{dy}{dt} = kx = ke^{-kt}
\]

\[
\Rightarrow \quad \int 1 \, dy = k \int e^{-kt} \, dt
\]

\[
\Rightarrow \quad y(t) = -e^{-kt} + D.
\]

The initial condition \( y(0) = 0 \) implies \( y(0) = 0 = -1 + D \) which implies \( D = 1 \). We have

\[
y(t) = 1 - e^{-kt}.
\]

**Solution (b):**  The plot is given below (see Figure 1). It can be clearly seen that as the reaction proceeds, the species \( A \) is used up and replaced by \( B \), as we would expect from simply looking at the reaction. The concentration of \( B \) approaches the initial concentration of the species \( A \).
Figure 1: The solutions $x(t) = e^{-kt}$ and $y(t) = 1 - e^{-kt}$ with $k = 1$. 