1 Series Tests

In the past two weeks, we have seen a number of different tests for determining whether a series $\sum_{n=1}^{\infty} c_n$ converges or diverges. Each of these tests has its own strengths and limitations. For convenience, we will list them out as a guide:

- **Geometric Series:** Any series which can be rearranged into the form
  $$\sum_{n=1}^{\infty} a r^{n-1}$$
  converges or diverges according to the rule
  $$\sum_{n=1}^{\infty} a r^{n-1} = \begin{cases} 
  \frac{a}{1-r}, & \text{for } |r| < 1 \\
  \text{does not exist,} & \text{for } |r| \geq 1.
  \end{cases}$$

- **$n^{th}$ Term Test:** A test for divergence only.

  **Proposition 1.1 ($n^{th}$ term test).** If $\lim_{n \to \infty} c_n \neq 0$ (or does not exist), then the series $\sum_{n=1}^{\infty} c_n$ diverges.

- **Integral Test:** Relates the series to an integral.

  **Proposition 1.2 (Integral test).** Suppose that the terms in the series $\sum_{n=1}^{\infty} c_n$ are denoted by $c_n = f(n)$, and $f(x)$ is a continuous, positive, decreasing function for $x \geq 1$. Then the series converges if and only if the improper integral $\int_{1}^{\infty} f(x) \, dx$ converges.

  This test is useful for series where it is immediately clear that if $n$ were replaced by $x$ everywhere, the resulting function would be immediately integrable, e.g. $1/(1 + n^2)$, $e^{-n}$, $1/n^p$, etc.

- **Comparison Test:** Relates the series to a series for which the convergence or divergence is already known.

  **Proposition 1.3 (Comparison Test).** If $0 \leq c_n \leq a_n$ for $n \geq 1$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} c_n$ converges. If $c_n \geq a_n \geq 0$ for $n \geq 0$ and $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} c_n$ diverges.
This test is limited by the fact that we have to guess whether the series converges or diverges before we can attempt to find a series which bounds it. This test is most useful (generally) for series which are closely related to $p$-series, e.g. $\frac{1}{\ln(n)n^2}$, $\frac{n}{(n^{1/2} + 1)}$, etc.

- **Limit Comparison Test:** Relates the series to a series which behaves similarly for large $n$.

**Proposition 1.4 (Limit Comparison Test).** If $0 \leq c_n$ and $0 < b_n$, and

$$\lim_{n \to \infty} \frac{c_n}{b_n} = \ell, \quad 0 < \ell < \infty,$$

then series $\sum_{n=1}^{\infty} c_n$ converges if $\sum_{n=1}^{\infty} b_n$ converges, and diverges if $\sum_{n=1}^{\infty} b_n$ diverges.

This test is one of the most applicable, and can be easily applied to complicated fractional limits where the behaviour as $n \to \infty$ is well defined, e.g. $\sqrt{n^2 - n + 2}/(3n^2 + n^{3/2} - 1)$, $(5^n - 4(2)^n - 1)/(2(3)^n - 2^n)$, etc.

- **Limit Ratio Test:** Relates the series to a geometric series through the ratio $c_{n+1}/c_n$.

**Proposition 1.5 (Limit Ratio Test).** Suppose that $c_n > 0$ and

$$\lim_{n \to \infty} \frac{c_{n+1}}{c_n} = L.$$

Then:

1. $\sum_{n=1}^{\infty} c_n$ converges if $L < 1$.
2. $\sum_{n=1}^{\infty} c_n$ diverges if $L > 1$ (or if $\lim_{n \to \infty} c_{n+1}/c_n = \infty$).
3. $\sum_{n=1}^{\infty} c_n$ may converge or diverge if $L = 1$.

This test is also useful in a wide range of applications. It is particular good at simplifying series involving factorial and other lengthy multiplicative terms (e.g. $1 \cdot 3 \cdot 5 \cdot \ldots (2n-1)$), as well as constants raised to the power $n$ (although these terms are also well handled by geometric series, the limit comparison test, and the limit root test).

- **Limit Root Test:** Relates the series to a geometric series through the root $\sqrt[n]{c_n}$.
Proposition 1.6 (Limit Root Test). Suppose that \( c_n \geq 0 \) and

\[
\lim_{n \to \infty} \sqrt[n]{c_n} = R.
\]

Then:

1. \( \sum_{n=1}^{\infty} c_n \) converges if \( R < 1 \).
2. \( \sum_{n=1}^{\infty} c_n \) diverges if \( R > 1 \) (or if \( \lim_{n \to \infty} \sqrt[n]{c_n} = \infty \)).
3. \( \sum_{n=1}^{\infty} c_n \) may converge or diverge if \( R = 1 \).

This test is well-suited for handling series which have terms raised to the power \( n \).

• Absolute Convergence Test: Relates the series to the series of absolute values.

Proposition 1.7 (Absolute Convergence Test). If the series \( \sum_{n=1}^{\infty} |c_n| \) converges, then the series \( \sum_{n=1}^{\infty} c_n \) converges.

This test is the first thing we try for series which contain both positive and negative terms \( c_n \). We can then attempt all of the tests above on the series \( \sum_{n=1}^{\infty} |c_n| \); however, this is a test for convergence only, and \( \sum_{n=1}^{\infty} |c_n| \) failing to converge does not necessarily imply the same of \( \sum_{n=1}^{\infty} c_n \).

• Alternating Series Test: This is our final test for alternating series which do not converge absolutely.

Proposition 1.8 (Alternating Series Test). An alternating series \( \sum_{n=1}^{\infty} (-1)^n c_n \) converges if the sequence of absolute values of the terms \( \{|c_n|\} \) is decreasing toward zero.

We notice that if this fails for an alternating series we can say something about the convergence of the series. If \( \{|c_n|\} \) does not tend toward zero, then certainly \( \{c_n\} \) does not tend toward zero, so that the series must diverge by the \( n^{th} \) term test.