1 Course Information

For a detailed breakdown of the course content and available resources, see the Course Syllabus (general course folder). Other relevant information for this section of MATH 118 is:

- **Instructor**: Matthew D. Johnston
- **Office**: MC 5126
- **Office hours**: 11:30-12:30 M, Tu
- **Tutorial**: 12:30-2:30 Tu (RCH 305)

(NOTE: the first tutorial will be Jan. 13)

2 Review

We will be studying integration for the first 3-4 weeks of this course. Topics which should already be familiar to you from MATH 116 include:

- **Indefinite integral (anti-differentiation):**
  \[ \int f'(x) \, dx = f(x) + C. \]

- **Definite integral (area under the curve):**
  \[ \int_a^b f(x) \, dx = \lim_{\|\Delta x\| \to 0} \sum_{i=1}^{n} f(x_i^*) \Delta x_i, \]

- **Fundamental Theorems of Calculus:**
  
  FTC #1: \[ \int_a^b f(x) \, dx = F(b) - F(a) \text{ where } F'(x) = f(x). \]
  FTC #2: \[ \frac{d}{dx} \int_a^x f(t) \, dt = f(x). \]
You should also be familiar with the various integral laws found on pg. 390 of Trim.

So we know integration “undoes” differentiation (FTC #1) and differentiation “undoes” integration (FTC #2). Furthermore, the Fundamental Theorem gives us a geometrical interpretation of the (somewhat abstract) process of anti-differentiation. We can find the area under a curve by evaluating the anti-derivative of a function at the end points and subtracting (at least for continuous functions). Throughout this course, we will see many applications of both of these interpretations of the integral (anti-derivative and area under the curve).

Knowing the relationship between derivatives and integrals, however, does not get us any closer to understanding how to integrate. We might first hope that we can develop methods of integration in the same way we developed them for differentiation. With differentiation, we were able to derive formulas (e.g. chain rule, product rule, quotient rule, etc.) which allowed us to differentiate complicated functions so long as we knew how to differentiate the individual components. For example, consider the function

\[ f(x) = \sin(x^2)e^{-x}. \]

Since we know how to differentiate \( x^2, \sin(x) \) and \( e^{-x} \), we can use the product and chain rules to derive, without much difficulty,

\[ \frac{d}{dx}f(x) = (2x \cos(x^2) - \sin(x^2))e^{-x}. \]

So we can “construct” the derivative by knowing the derivatives of the component pieces — this is a very nice property of differentiation!

We may naively hope that integration follows the same “constructive” pattern — i.e. that integrating individual components of a function allows us to “construct” the integral of the whole. For example, we might hope that knowing how to integrate \( x^2, \sin(x) \) and \( e^{-x} \) will allow us to solve

\[ \int \sin(x^2)e^{-x} \, dx. \]

Unfortunately, this is not the case — no general formulas for constructing integrals exists, even when the integrals of the components of the function are known. In fact, for some functions, no closed-form integral exists at all! (This function is an example of this. These cases can be handled by numerical integration.)

As a result, integration can be more difficult than differentiation. It often feels like we do not know how to begin. However, we are not completely out
of luck — far from it. In fact, we already know an entire class of functions for which we can easily find the integral — any function which is the derivative of another function!

Since integration is really anti-differentiation, any function we recognize as the derivative of another function can be easily integrated. For example, we have

\[
\frac{d}{dx} \left( \frac{x^2}{2} \right) = x \quad \Rightarrow \quad \int x \, dx = \frac{x^2}{2} + C
\]

\[
\frac{d}{dx} \sin(x) = \cos(x) \quad \Rightarrow \quad \int \cos(x) \, dx = \sin(x) + C
\]

\[
\frac{d}{dx} e^x = e^x \quad \Rightarrow \quad \int e^x \, dx = e^x + C
\]

(NOTE: for a more complete list, see pg. 338-339 of Trim.)

The rest of the major integration techniques essentially amount to manipulating equations to get them into a form where we recognize the function to be integrated as the derivative of another function. The first of these techniques we will talk about is Integration By Substitution (Section 8.1, Trim).

3 Integration By Substitution

One principle technique of integration is integration by substitution. The central idea is that, while a function may not be readily integrable with respect to its current variable, there may be another variable we can integrate with respect to for which the integration step is much easier. Integration by substitution is justified by the chain rule — we won’t cover this in detail.

The algorithm for integration by substitution is:

1. Choose your new variable (say, \( u = g(x) \))
2. Replace the old variable with the new (including \( dx \), etc.)
3. Integrate with respect to the new variable
4. Replace the new variable with the old
5. Check the answer! (Take the derivative.)
There are a few things to note here. The first is that the new variable is only needed for the integration step — the final answer should be stated with respect to the old variable. Remember to switch back!

Another thing to remember is that the quantity to be integrated must be stated entirely in terms of the new variable — all instances of the old variable must be replaced with the new. Under no circumstances may you integrate with respect to mixed variables. If you are unable to entirely replace the old variable with the new, you have probably chosen a poor change of variable and should try something else.

**Example 1:**

Consider the integral
\[
\int x\sqrt{2x-1} \, dx.
\]

If you can immediately recognize a function whose derivative is \(x\sqrt{2x-1}\), you are a better mathematician than I am. Instead, we should try a substitution, but what should we pick? The thing to note here is that our real trouble point is the square root, since it does not distribute across \(2x-1\). The integral would be easier if the root were with respect to a single variable, so let’s choose
\[
u = 2x - 1.
\]

Remember, however, that we need to replace all instances of \(x\) from original integral. Since \(u\) depends on \(x\), we can evaluate
\[
\frac{du}{dx} = 2 \implies dx = \frac{1}{2} du.
\]

All that remains is to substitute is the \(x\), which can be found by rearranging the original substitution to give
\[
x = \frac{1}{2}(u + 1).
\]

We have everything we need, so let’s plug it into the original integral:
\[
\int \frac{1}{2}(u + 1)u^{1/2} \frac{1}{2} du = \frac{1}{4} \int \left[u^{3/2} + u^{1/2}\right] du.
\]

Everything under the integral is in terms of the new variable \(u\) so we can integrate over \(u\) to get
\[
\frac{1}{4} \left[\frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right] + C = \frac{1}{10}u^{5/2} + \frac{1}{6}u^{3/2} + C.
\]
That was surprisingly painless, but we are not quite done. The original expression was in terms of \(x\), so that is how we should state the answer. We only needed the new variable to simplify the integration step. Using the original substitution, our final answer is
\[
\frac{1}{10}(2x-1)^{5/2} + \frac{1}{6}(2x-1)^{3/2} + C.
\]

But how do we know this is correct? We don’t, after all, necessarily have a pre-set idea what the integral should look like. We can, however, simply take the derivative and hope we arrive at the original expression. Since taking derivatives is easy to do, it usually does not take very long to check our answer. This should be a common practice!

For this example, we have
\[
\frac{d}{dx} \left[ \frac{1}{10}(2x-1)^{5/2} + \frac{1}{6}(2x-1)^{3/2} + C \right] = 2 \left( \frac{1}{10} \right) \left( \frac{5}{2} \right) (2x-1)^{3/2} + 2 \left( \frac{1}{6} \right) \left( \frac{3}{2} \right) (2x-1)^{1/2} = \frac{1}{2}(2x-1)^{3/2} + \frac{1}{2}(2x-1)^{1/2} = \frac{1}{2}(2x-1)^{1/2} (2x-1+1) = x(2x-1)^{1/2}.
\]

So our answer is correct.

**Example 2:**

Integration by substitution can also be applied to definite integrals; however, we need to keep track of how the bounds of integration change as a result of the substitution. After our change of variable, we have two options in this regard:

1. Find the new bounds, integrate, and solve using the new bounds directly.

2. Ignore the bounds, integrate, replace new variable with old and solve using the old bounds.

Which technique you use is up to you, but you MUST be clear what the bounds are. If you mix up the bounds for the variables, you will likely end up with a spectacularly wrong answer.
Consider the definite integral
\[ \int_0^9 \frac{(1 + \sqrt{x})^{1/2}}{\sqrt{x}} \, dx. \]

Again, the problem with integrating this directly is the root, so we try the substitution
\[ u = 1 + \sqrt{x}. \]

This readily leads to
\[ \frac{du}{dx} = \frac{1}{2\sqrt{x}} \implies dx = 2\sqrt{x} \, du. \]

However, we still need to consider the bounds. When \( x = 0 \) we have \( u = 1 \) and when \( x = 9 \) we have \( u = 4 \), so when we substitute \( u \) for \( x \), we are now integrating from 1 to 4.

We have
\[
2 \int_1^4 u^{1/2} \, du = \left[ \frac{4}{3} u^{3/2} \right]_1^4 = \frac{4}{3} \left( 4^{3/2} - 1^{3/2} \right) = \frac{28}{3}.
\]

We could have neglected the change in the bounds but we would have had to have remembered to change back to the original variable after integrating before evaluating at the bounds. For example,
\[
\int_0^9 \frac{(1 + \sqrt{x})^{1/2}}{\sqrt{x}} \, dx = 2 \int_{x=0}^{x=9} u^{1/2} \, du = \left[ \frac{4}{3} u^{3/2} \right]_{x=0}^{x=9} = \frac{4}{3} \left( 1 + \sqrt{9} \right)^{3/2} - \frac{4}{3} = \frac{28}{3}.
\]