1 Improper Integrals

That the definite integral could be evaluated by substituting the endpoints of the region into the antiderivative and subtracting was justified by the very important assumption that the function to be integrated was continuous in the intermediate region. When this assumption is not justified, or when the bounds of integration are not finite, we have to take an alternative approach.

Consider
\[ \int_{-1}^{1} \frac{1}{x^2} \, dx. \]

For a moment, let’s forget what I have just told you and assume we are justified in evaluating this as we have been. We have
\[ \int_{-1}^{1} \frac{1}{x^2} \, dx = -\left[ \frac{1}{x} \right]_{-1}^{1} = -[1 + 1] = -2. \]

If we had no intuition regarding what the definite integral should be (the area under the curve), we might very well move on not realizing we have made a severe miscalculation. However, we do have some intuition, and we call tell by the looking at the graph of \( f(x) = \frac{1}{x^2} \) that there is no conceivable way the area bound \( x = -1 \) and \( x = 1 \) could be negative, let alone attain the value \(-2\) (see Figure 1).

But what have we done wrong? This example is subtly different than the previous examples we have seen in that there is a discontinuity at \( x = 0 \). It turns out that this is exactly the problem.

Our next question is how we might approach rectifying this problem. We notice that for any interval wholly to the left of \( x = 0 \) (say, \( x = -1 \) to \( x = a \) where \( a < 0 \)), we can apply the standard formula, and the same applies for any interval wholly to the right of \( x = 0 \) as well (\( x = b \) to \( x = 1 \) where \( b > 0 \)). We are allowed to break integrals into parts like this, so we will consider one integral on the right and one of the left. But how can we manipulate the left and right integrals to capture the whole area we are looking for? In fact, the answer is perhaps the simplest tool we have available to us: we simply
Figure 1: Area under the curve for $f(x) = \frac{1}{x^2}$.

take the limit!

$$\int_{-1}^{1} \frac{1}{x^2} \, dx$$

$$= \lim_{a \to 0^-} \int_{-1}^{a} \frac{1}{x^2} \, dx + \lim_{b \to 0^+} \int_{b}^{1} \frac{1}{x^2} \, dx$$

$$= \lim_{a \to 0^-} \left[ -\frac{1}{x} \right]_{-1}^{a} + \lim_{b \to 0^+} \left[ -\frac{1}{x} \right]_{b}^{1}$$

$$= \lim_{a \to 0^-} \left[ -\frac{1}{a} + 1 \right] + \lim_{b \to 0^+} \left[ -1 + \frac{1}{b} \right] = \infty.$$

Something else of interest has happened here. We have not seen functions with unbounded definite integrals before; however, we should not consider them unexpected. Around $x = 0$, $f(x) = 1/x^2$ goes off to infinity very quickly. This intuition will not always hold (the area under the curve can be finite even when the region being integrated is unbounded!), but often times it will.
Consider now being asked to evaluate

\[ \int_0^1 \ln(x) \, dx. \]

Again, we cannot evaluate this directly because \( \ln(x) \) becomes unbounded (limiting towards \(-\infty\)) as \( x \) approaches 0 from the right. We consider instead the limit of integrals with bounds approaching \( x = 0 \).

\[ \int_0^1 \ln(x) \, dx = \lim_{a \to 0^-} \int_a^1 \ln(x) \, dx = \lim_{a \to 0^-} [x \ln(x) - x]_a^1 = \lim_{a \to 0^-} [(1) \ln(1) - 1 - a \ln(a) - a] = -1 - \lim_{a \to 0^-} a \ln(a). \]

The remaining limit is of the indeterminant form \( 0 \cdot \infty \). We rearrange and solve using l’Hopital’s rule

\[ \lim_{a \to 0^-} \frac{\ln(a)}{\frac{1}{a}} = - \lim_{a \to 0^-} \frac{a}{\ln(a)} = - \lim_{a \to 0^-} a = 0. \]

This implies that

\[ \int_0^1 \ln(x) \, dx = -1. \]

We should notice here that, unlike the previous example, the area under the curve (or rather, over the curve in this case) is finite even though the region being integrated is unbounded.

The concept of taking limits for the bounds of integration is applicable for another class of integrals as well. Consider being asked to evaluate

\[ \int_1^\infty \frac{1}{x} \, dx. \]

We cannot simply integrate and plug in the bound \( x = \infty \); however, we can integrate over \( x \) from 1 to some value \( a > 1 \) and then take the limit as \( a \to \infty \). Let’s do this.
\[ \int_{1}^{\infty} \frac{1}{x} \, dx = \lim_{a \to \infty} \int_{1}^{a} \frac{1}{x} \, dx = \lim_{a \to \infty} [\ln(x)]_{1}^{a} = \lim_{a \to \infty} [\ln(a) - \ln(1)] = \infty. \]

Now consider evaluating

\[ \int_{1}^{\infty} \frac{1}{x^n} \, dx, \quad n > 1. \]

We apply the same technique to get

\[ \int_{1}^{\infty} \frac{1}{x^n} \, dx = \lim_{a \to \infty} \int_{1}^{a} \frac{1}{x^n} \, dx = \frac{1}{1 - n} \lim_{a \to \infty} \left[ \frac{1}{x^{n-1}} \right]_{1}^{a} = \frac{1}{1 - n} \left[ \lim_{a \to \infty} \frac{1}{a^{n-1}} - 1 \right]. \]

Since \( n - 1 > 0 \) by assumption we have

\[ \lim_{a \to \infty} a^{n-1} = \infty \quad \text{so that} \quad \lim_{a \to \infty} \frac{1}{a^{n-1}} = 0. \]

Putting everything together, we have

\[ \int_{1}^{\infty} \frac{1}{x^n} \, dx = \frac{1}{n - 1}, \quad n > 1. \]

We notice a few things here. We see that \( n = 1 \) is an important point in some sense in that for \( n = 1 \) the integral is unbounded, but for any value of \( n \) greater than one, it is finite. We also see that, as we might expect,

\[ \lim_{n \to 1^+} \frac{1}{x^n} \, dx = \lim_{n \to 1^+} \frac{1}{n - 1} = \infty = \int_{1}^{\infty} \frac{1}{x} \, dx. \]