1 Polar Areas

We recall from Cartesian coordinates that we could calculate the area under the curve by taking Riemann sums. We divided the region into subregions, approximated the area over that subregion by a rectangle, added all of the regions up, and then took the limit as the width of the subregions went to zero. Furthermore, we were able to show that this process was equivalent to the process of taking the definite integral!

We can carry out a very similar procedure in polar coordinates; however, rather than approximating our areas with rectangles we will be approximating them by wedges of a circle (see Figure 1). Otherwise, the same exact principle applies: we divide the region into subregions (of $\theta$), approximate the integral within the subregions by wedges, add these approximates up, and then take the limit as the width of the subregions goes to zero.

![Diagram showing wedge approximations](image)

Figure 1: We can approximate the area of a region as the sum of wedges. ($t_0 = \theta_0, t_1 = \theta_1$, etc.)
The question remains as to how we can approximate the areas of the wedges, $A_i$. We will start by considering the whole circle associated with a particular wedge. The area of the circle associated with the region $A_i$ is

$$\text{Area} = \pi r^2 = \pi f(\theta_i)^2.$$

Now consider the wedge between, say, $\theta_i$ and $\theta_{i+1}$. This wedge represents a fraction of the whole area of the circle, but how do we quantify the fraction? We notice that we would recover the area for the entire circle if we took $\theta_i = 0$ and $\theta_{i+1} = 2\pi$, so the fraction we need is $(\theta_{i+1} - \theta_i)/(2\pi)$. This gives

$$\text{Area of wedge } A_i = \frac{\theta_{i+1} - \theta_i}{2\pi} \pi r^2 = \frac{1}{2}(\theta_{i+1} - \theta_i)f(\theta_i)^2.$$

So we have a formula for the area of our wedge, which approximates the actual area bound between the two angles. If we divide the region in $n$ wedges with widths $\Delta \theta_i = \theta_{i+1} - \theta_i$ and take the sum, we have

$$\sum_{i=1}^{n} A_i = \sum_{i=1}^{n} \frac{1}{2}[f(\theta_i)]^2 \Delta \theta_i.$$

We now take the limit as the widths of the regions go to zero to get

$$\text{Area} = \lim_{\|\Delta \theta_i\| \to 0} \sum_{i=1}^{n} \frac{1}{2}[f(\theta_i)]^2 \Delta \theta_i.$$

We recognize the right-hand side as the integral of $(1/2)[f(\theta)]^2$ over $\theta$. We take the bounds on $\theta$ to be $\alpha$ and $\beta$ to get

$$\text{Area} = \frac{1}{2} \int_{\alpha}^{\beta} [f(\theta)]^2 \, d\theta.$$

We can do all of the tricks we could do with definite integrals in Cartesian coordinates: measure areas between curves, divide the integral into separate integrals to be calculated separately, etc. We must remember, however, that we are dividing things in terms of $\theta$ rather than $x$.

**Example 1:**

Find the area of one of the “petals” of the function

$$r = \sin(2\theta).$$
We might as well consider the petal bound by \( \theta = 0 \) and \( \theta = \frac{\pi}{2} \) (see Figure 2). We have

\[
\text{Area} = \frac{1}{2} \int_{\alpha}^{\beta} [f(\theta)]^2 \, d\theta
= \frac{1}{2} \int_{0}^{\frac{\pi}{2}} \sin^2(2\theta) \, d\theta
= \frac{1}{4} \int_{0}^{\frac{\pi}{2}} [1 - \cos(4\theta)] \, d\theta
= \frac{1}{4} \left[ \theta - \frac{\sin(4\theta)}{4} \right]_{0}^{\frac{\pi}{2}}
= \frac{\pi}{8}.
\]

Figure 2: The area of one “petal” of the polar equation \( r = \sin(2\theta) \). The area is \( \frac{\pi}{8} \).

**Example 2:**

Find the area bound by the curve

\[ r^2 = -\cos(\theta). \]

We notice immediate that, since \( r^2 \) is always positive, this relation is only defined when \( \cos(\theta) \) is negative, that is to say, in the region \( \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} \).
We solve for $r$ to get

$$r = \pm \sqrt{-\cos(\theta)}, \quad \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}.$$ 

The positive root gives a bound region to the left of the $y$-axis; the negative root gives the reflection of that region across the $y$-axis (see Figure 3). Since the two regions have the same area, it is sufficient to calculate the area for one region and then double it. We will take the positive root to get

$$\text{Area} = (2) \frac{1}{2} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left( \sqrt{-\cos(\theta)} \right)^2 d\theta$$

$$= - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos(\theta) \, d\theta$$

$$= - \left[ \sin(\theta) \right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = 2.$$

Figure 3: The area bound by the polar relation $r^2 = -\cos(\theta)$. The area is 2 (or 1 for each half!).

**Example 3:**

Find the area outside the curve $r = 3$ but inside the curve $r = 6\sin(\theta)$.
Our first task is to figure out what the graphs of \( r = 3 \) and \( r = 6 \sin(\theta) \) look like. In this case, this is easiest handled by converting to Cartesian coordinates. We immediately recognize \( r = 3 \) as the giving the graph of the circle of radius 3 centred at \((0, 0)\), i.e.

\[
x^2 + y^2 = 9.
\]

The second is a little trickier, but we use our identities \( r = \sqrt{x^2 + y^2} \) and \( \sin(\theta) = y/\sqrt{x^2 + y^2} \) to get

\[
\sqrt{x^2 + y^2} = 6 \frac{y}{\sqrt{x^2 + y^2}} \quad \Rightarrow \quad x^2 + y^2 - 6y = 0
\]

\[
\Rightarrow \quad x^2 + (y - 3)^2 = 9.
\]

This is the equation of the circle of radius three centred at \((0, 3)\). We can now clearly see the area we are being asked to calculate (see Figure 4).

![Figure 4: The area outside the circle \( r = 3 \) but within the circle \( r = 6 \sin(\theta) \).](image)

We need to figure out our bounds of integration. We can clearly see that for each “wedge” \( \Delta \theta \), we can calculate the area by taking the upper circle \( r = 6 \sin(\theta) \) minus the lower \( r = 3 \); however, we need to calculate the intersection points in order to find the bounds for \( \theta \). As with Cartesian coordinates, we can do this by setting the two polar equations equal to one another. Since points have multiple representations in polar coordinates,
however, we will have to check with the graph to see if our results are reasonable or not. We have

\[ 6 \sin(\theta) = 3 \implies \theta = \arcsin \left( \frac{1}{2} \right). \]

From the graph, the two angles of interest to us are \( \theta = \frac{\pi}{6} \) and \( \theta = \frac{5\pi}{6} \), so these are our bounds of integration.

We have

\[
\text{Area} = \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} [6 \sin(\theta)]^2 - [3]^2 \, d\theta \\
= \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} [36 \sin^2(\theta) - 9] \, d\theta \\
= \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} [18 - 18 \cos(2\theta) - 9] \, d\theta \\
= \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} [9 - 18 \cos(2\theta)] \, d\theta \\
= \frac{1}{2} [9\theta - 9 \sin(2\theta)]_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \\
= \frac{9}{2} \left[ \frac{5\pi}{6} - \frac{\pi}{6} - \sin \left( \frac{5\pi}{3} \right) + \sin \left( \frac{\pi}{3} \right) \right] \\
= \frac{9}{2} \left[ \frac{2\pi}{3} + \sqrt{3} \right] \\
= 3\pi + \frac{9\sqrt{3}}{2} \approx 17.219.
\]

We notice that overall area of the upper circle is \( \pi r^2 = \pi (3)^2 \approx 28.174 \). It seems reasonable that the area remaining over subtracting the intersection with the lower circle is on the order of 17 units.

**Example 4:**

Find the area bound by the curves \( r = \sin(\theta) \) and \( r = \cos(\theta) \).

We start by visualizing what the curves look like. In this case, this is easiest handled by transforming into Cartesian coordinates. We have

\[ r = \sin(\theta) \implies \sqrt{x^2 + y^2} = \frac{y}{\sqrt{x^2 + y^2}} \]
\[ \Rightarrow x^2 + y^2 - y = 0 \quad \Rightarrow \quad x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}. \]

We also have
\[ r = \cos(\theta) \quad \Rightarrow \quad \sqrt{x^2 + y^2} = \frac{x}{\sqrt{x^2 + y^2}} \]
\[ \Rightarrow x^2 - x + y^2 = 0 \quad \Rightarrow \quad \left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4}. \]

These are the circles of radius 1/2 centred at (0, 1/2) and (1/2, 0) respectively. We can clearly see the area bound by the two curves (see Figure 5).

Figure 5: The area bound by the curves \( y = \cos(\theta) \) and \( r = \cos(\theta) \).

We need to consider our bounds of integration. If we imagine arrays extending from the origin, we can see that as \( \theta \) rotates through the region \([0, \pi/2]\) there is a point where the upper bound (the \( r \)) changes from being relative to the one circle and switches to the other. We need to know what this point is, so we look for points of intersection of the curves by setting the equations equal.

\[ \sin(\theta) = \cos(\theta) \quad \Rightarrow \quad \tan(\theta) = 1 \quad \Rightarrow \quad \theta = \frac{\pi}{4}. \]

This implies that in the region \( \theta = 0 \) to \( \theta = \pi/4 \) we are integrating to \( r = \sin(\theta) \), and in the region \( \theta = \pi/4 \) to \( \theta = \pi/2 \) we are integrating to
\( r = \cos(\theta) \). (We could also notice the areas are identical, and thus we only need to compute one, but we will follow all of the steps.) Applying the formula, we have

\[
\text{Area} = \frac{1}{2} \int_0^{\pi/2} \sin^2(\theta) \, d\theta + \frac{1}{2} \int_{\pi/4}^{\pi/2} \cos^2(\theta) \, d\theta
\]

\[
= \frac{1}{4} \int_0^{\pi/4} [1 - \cos(2\theta)] \, d\theta + \frac{1}{4} \int_{\pi/4}^{\pi/2} [1 + \cos(2\theta)] \, d\theta
\]

\[
= \frac{1}{4} \left[ \theta - \frac{\sin(2\theta)}{2} \right]_0^{\pi/4} + \frac{1}{4} \left[ \theta + \frac{\sin(2\theta)}{2} \right]_{\pi/4}^{\pi/2}
\]

\[
= \frac{\pi}{8} - \frac{1}{4} \approx 0.143.
\]