1 Series

1.1 Limit Ratio Test

Consider the convergence of the series
\[ \sum_{n=1}^{\infty} \frac{1}{n!}. \]

This can be handled by the comparison test, taking an appropriate choice of bounding function. We will, however, use an alternative intuition. We can relate successive terms \( c_{n+1} \) and \( c_n \) very easily: we have
\[ c_{n+1} = \frac{1}{n} c_n \quad \text{or} \quad \frac{c_{n+1}}{c_n} = \frac{1}{n}. \]

There was another series we could relate in this fashion, which was the geometric series where the difference between the successive terms given by the relationship
\[ c_{n+1} = r c_n. \]

We recall that geometric series converged if \( |r| < 1 \) and diverged if \( |r| \geq 1 \).

But how do we relate our factor \( 1/n \) to the geometric series multiplicative factor \( r \)? For geometric series, \( r \) was fixed for all values of \( n \), so how do we know which value of \( n \) to pick to get our \( r \)? We recall that the convergence or divergence of series is really dependent only on the large values of \( n \), so that we can can take the limit as \( n \to \infty \). Since \( 1/n \) goes to 0 as \( n \to \infty \), we suspect this series in convergent. In fact, this is justified by the following result:

**Proposition 1.1** (Limit Ratio Test). *Suppose that \( c_n > 0 \) and
\[ \lim_{n \to \infty} \frac{c_{n+1}}{c_n} = L. \]

Then:
1. $\sum_{n=1}^{\infty} c_n$ converges if $L < 1$.

2. $\sum_{n=1}^{\infty} c_n$ diverges if $L > 1$ (or if $\lim_{n \to \infty} c_{n+1}/c_n = \infty$).

3. $\sum_{n=1}^{\infty} c_n$ may converge or diverge if $L = 1$.

In this case, we have $\lim_{n \to \infty} c_{n+1}/c_n = \lim_{n \to \infty} 1/n = 0$ so the series converges.

**Justification:** We can handle this by cases. In the first two cases, we will use the formal definition of a limit to construct a geometric sequence which bounds the original series (from above in the first case, and below in the second)—we will then use the comparison test.

From the formal definition of a limit, we have that $\lim_{n \to \infty} c_{n+1}/c_n = L$ implies that, for all $\epsilon > 0$ there exists an $N \geq 1$ such that

$$\left| \frac{c_{n+1}}{c_n} - L \right| < \epsilon \implies (L - \epsilon) c_n < c_{n+1} < (L + \epsilon) c_n, \quad \text{for } n \geq N.$$ 

**Case 1:** We will use the right-hand part of the above inequality. We notice something very important about the term $L + \epsilon$. We had freedom in how we select $\epsilon > 0$—specifically, we can choose it to be as small as required. This means that, if $L < 1$ then we can select $L + \epsilon$ such that $L < L + \epsilon < 1$! In general, our value $N \geq 1$ will depend on this choice of $\epsilon$; regardless of its value, we can see that

$$c_{n+1} < (L + \epsilon)c_n$$
$$c_{n+2} < (L + \epsilon)c_{n+1} < (L + \epsilon)^2 c_n$$
$$c_{n+3} < \cdots$$

so that we have

$$c_N + c_{N+1} + c_{N+2} + \cdots < c_N + (L+\epsilon)c_N + (L+\epsilon)^2 c_N + \cdots = \sum_{k=1}^{\infty} c_N(L+\epsilon)^{k-1}.$$ 

In other words, the series for $n \geq N$ is bound from above by a geometric series with $a = c_N$ and $r = (L + \epsilon) < 1$—this is a convergent series! Since the convergence of the series depends on the values for large $n$, we have that the series $\sum_{n=1}^{\infty} c_n$ converges by the comparison test.
Case 2: Now we will use the left-hand part of the above inequality. We notice that for $L > 1$ we can pick $\epsilon > 0$ small enough so that $1 < L - \epsilon < L$. Our value $N \geq 1$ will depend on this choice of $\epsilon$, and we can see that

\[ c_{N+1} > (L - \epsilon)c_N \]
\[ c_{N+2} > (L - \epsilon)^2c_N \]
\[ c_{N+3} > \cdots \]

so that we have

\[ c_N + c_{N+1} + c_{N+2} + \cdots > c_N + (L - \epsilon)c_N + (L - \epsilon)^2c_N + \cdots = \sum_{k=1}^{\infty} c_N(L - \epsilon)^{k-1}. \]

We can see that the right-hand side is given by a geometric series with $a = c_N$ and $r = L - \epsilon > 1$—by the comparison test, we have that the original series $\sum_{n=1}^{\infty} c_n$ diverges since the geometric series does.

Case 3: The convergence of series with $L = 1$ is not determinable by this test. For instance, we know from the integral test that $\sum_{n=1}^{\infty} 1/n$ diverges while $\sum_{n=1}^{\infty} 1/n^2$ converges; however, both series satisfy $\lim_{n \to \infty} c_{n+1}/c_n = 1$.

Example 1:

Determine whether the series

\[ \sum_{n=1}^{\infty} \frac{e^n}{n^4} \]

converges or diverges.

We apply the limit ratio test to get

\[ \lim_{n \to \infty} \frac{c_{n+1}}{c_n} = \lim_{n \to \infty} \frac{e^{n+1}}{(n+1)^4} \cdot \frac{n^4}{e^n} = \lim_{n \to \infty} e \left( \frac{n}{n+1} \right)^4 = e > 1. \]

By the limit ratio test, the series $\sum_{n=1}^{\infty} e^n/n^4$ diverges.

Example 2:

Determine whether the series

\[ \sum_{n=1}^{\infty} \frac{(n-1)(n-2)}{n^{2n}} \]
converges or diverges.

We apply the limit ratio test to get

$$\lim_{n \to \infty} \frac{c_{n+1}}{c_n} = \lim_{n \to \infty} \frac{n(n-1)}{(n+1)^2 2^{n+1}} \cdot \frac{n^2 2^n}{(n-1)(n-2)}$$

$$= \lim_{n \to \infty} \frac{1}{2} \left( \frac{n}{n-2} \right) \left( \frac{n}{n+1} \right)^2$$

$$= \lim_{n \to \infty} \frac{1}{2} \left( \frac{1}{1 - \frac{2}{n}} \right) \left( \frac{1}{1 + \frac{1}{n}} \right)^2 \approx \frac{1}{2} < 1.$$

By the limit ratio test, the series $\sum_{n=1}^{\infty} \frac{n(n-1)(n-2)}{n^2 2^n}$ converges.

### 1.2 Limit Root Test

Consider the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^n}.$$

Once again, this can be handled by the comparison test but we will use a different intuition. We notice that the terms of the geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ satisfy

$$c_n = ar^{n-1} = \left( \frac{a}{r} \right) r^n$$

so that

$$\sqrt[n]{c_n} = \left( \frac{a}{r} \right)^{1/n} r \approx r \quad \text{for } n \text{ large}$$

(since $(a/r)^{1/n} \to 1$ as $n \to \infty$). Since the convergence or divergence of a series depends on the values for large $n$ we suspect that since

$$\lim_{n \to \infty} \sqrt[n]{c_n} = \lim_{n \to \infty} \frac{1}{n} = 0 < 1$$

the series behaves like a convergent geometric series for large $n$, and therefore converges. This is justified by the following result.

**Proposition 1.2 (Limit Root Test).** Suppose that $c_n \geq 0$ and

$$\lim_{n \to \infty} \sqrt[n]{c_n} = R.$$

Then:
1. \( \sum_{n=1}^{\infty} c_n \) converges if \( R < 1 \).

2. \( \sum_{n=1}^{\infty} c_n \) diverges if \( R > 1 \) \( \text{or if } \lim_{n \to \infty} \sqrt[n]{c_n} = \infty \).

3. \( \sum_{n=1}^{\infty} c_n \) may converge or diverge if \( R = 1 \).

**Justification:**

The justification for this result is very similar to the justification for the limit ratio test. We will use the definition of a limit to generate a geometric sequence which we will compare to the original series.

From the definition of a limit, we have that \( \lim_{n \to \infty} \sqrt[n]{c_n} = R \) implies that for every \( \epsilon > 0 \) there exists an \( N \geq 1 \) such that

\[
|\sqrt[n]{c_n} - R| < \epsilon \quad \Rightarrow \quad (R - \epsilon)^n < c_n < (R + \epsilon)^n, \quad \text{for } n \geq N.
\]

**Case 1:** Similarly to the limit ratio test, for \( R < 1 \) we consider the right-hand side of the above inequality. Since we can take \( \epsilon > 0 \) arbitrarily small, we can select it so that \( R < R + \epsilon < 1 \). For the corresponding \( N \geq 1 \) we have that

\[
\begin{align*}
c_N &< (R + \epsilon)^N \\
c_{N+1} &< (R + \epsilon)^{N+1} \\
c_{N+2} &< \cdots
\end{align*}
\]

so that we have

\[
c_N + c_{N+1} + c_{N+2} + \cdots < (R + \epsilon)^N + (R + \epsilon)^{N+1} + \cdots = \sum_{k=1}^{\infty} (R + \epsilon)^N (R + \epsilon)^{k-1}.
\]

Since \( R + \epsilon < 1 \), this is a convergent geometric series, which implies by the comparison test that \( \sum_{n=N}^{\infty} c_n \) converges, which in turn implies that \( \sum_{n=1}^{\infty} c_n \) converges.

**Case 2:** For \( R > 1 \) we consider the left-hand side of the above inequality. Since we can take \( \epsilon > 0 \) arbitrarily small, we can select it so that \( 1 < R - \epsilon < R \). For the corresponding \( N \geq 1 \) we have that

\[
\begin{align*}
c_N &> (R - \epsilon)^N \\
c_{N+1} &> (R + \epsilon)^{N+1} \\
c_{N+2} &> \cdots
\end{align*}
\]

so that we have

\[
c_N + c_{N+1} + c_{N+2} + \cdots > (R + \epsilon)^N + (R + \epsilon)^{N+1} + \cdots = \sum_{k=1}^{\infty} (R + \epsilon)^N (R + \epsilon)^{k-1}.
\]
Since $R + \epsilon > 1$, this is a divergent geometric series, which implies by the comparison test that $\sum_{n=N}^{\infty} c_n$ diverges, which in turns implies that $\sum_{n=1}^{\infty} c_n$ diverges.

**Case 3:** Again, the series $\sum_{n=1}^{\infty} 1/n$ and $\sum_{n=1}^{\infty} 1/n^2$ suffice to show that $R = 1$ is insufficient to determine the convergence or divergence of a series.

**Note:** One particularly common recurring limit that arises from the limit root test is

$$\lim_{n \to \infty} n^{1/n}.$$ 

There are two competing behaviours here: the $n$ in the base wants to take the limit to infinity (e.g. $\lim_{n \to \infty} n^{1/a} = \infty$ for $a > 0$) while the $n$ in the exponent wants to reduce things to one (e.g. $\lim_{n \to \infty} a^{1/n} = 1$ for $a > 0$). We need some way to determine which behaviour wins out in the limit of the sequence.

We can solve this equation by considering the limit of the logarithm of the series. We have

$$\lim_{n \to \infty} \ln(n^{1/n}) = \lim_{n \to \infty} \frac{\ln(n)}{n} = \lim_{n \to \infty} \frac{1}{n} \quad \text{(l’Hopital’s rule)}$$

$$= 0.$$ 

This was the limit for $\ln(c_n)$. In order to determine the limit for $c_n$ we exponentiate both sides to get

$$\lim_{n \to \infty} n^{1/n} = e^0 = 1.$$ 

This result will be required frequently. We can also generalize to general exponents $a/n$ where $a \neq 0$ so that

$$\lim_{n \to \infty} n^{a/n} = 1.$$ 

**Example 1:**

Determine whether the series

$$\sum_{n=1}^{\infty} \frac{n}{(\ln(n))^{n}}$$
converges or diverges.

We apply the limit root test to get

$$\lim_{n \to \infty} \sqrt[n]{c_n} = \lim_{n \to \infty} \frac{n^{1/n}}{\ln(n)} = 0 < 1.$$ 

By the limit root test, the series converges.

**Example 2:**

Determine whether the series

$$\sum_{n=1}^{\infty} \frac{2^n + n^2 3^n}{4^n}$$

converges or diverges.

We have

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n + \sum_{n=1}^{\infty} n^2 \left(\frac{3}{4}\right)^n.$$ 

The first series is a geometric series with $r = 1/2$, so it converges. The second we can handle by using the limit root test. We have

$$\lim_{n \to \infty} \sqrt[n]{c_n} = \lim_{n \to \infty} \frac{3}{4} n^{2/n} = \frac{3}{4} < 1.$$ 

By the limit root test, the second series converges. Since both series converge, the overall series converges.