1 Absolute Convergence

All of the series $\sum_{n=1}^{\infty} c_n$ we have dealt with so far have had nonnegative terms $c_n$. This begs the question of how we determine the convergence/divergence of series with an infinite number of both positive and negative terms.

To handle this case, we introduce the concept of absolute convergence. A series is said to be absolutely convergent if

$$\sum_{n=1}^{\infty} |c_n|$$

converges. We notice that the series of absolute values is nonnegative and consequently all of our convergence results apply to this series. This fact is made useful by the following result:

**Proposition 1.1** (Absolute Convergence Test). *If the series $\sum_{n=1}^{\infty} |c_n|$ converges, then the series $\sum_{n=1}^{\infty} c_n$ converges.*

**Justification:** We will divide the series $\sum_{n=1}^{\infty} c_n$ into positive and negative components as follows:

Let $\{p_n\}$ be the sequence of $c_n$ with positive values

Let $\{n_n\}$ be the sequence of absolute values of $c_n$ with negative values.

It is clear from these definitions that

$$\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} p_n - \sum_{n=1}^{\infty} n_n.$$ 

What we need to show is that $\sum_{n=1}^{\infty} |c_n|$ converging implies that $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} n_n$ converge.

Given this decomposition of positive and negative terms in $\{c_n\}$ we have that

$$\sum_{n=1}^{\infty} |c_n| = \sum_{n=1}^{\infty} p_n + \sum_{n=1}^{\infty} n_n = L < \infty$$
since $\sum_{n=1}^{\infty} |c_n|$ converges. Since both of the partial sums $\sum_{k=1}^{n} p_k$ and $\sum_{k=1}^{n} n_k$ are increasing and bounded (by $L$) we know that they approach a limit, that is to say $\lim_{n \to \infty} \sum_{k=1}^{n} p_k = \sum_{n=1}^{\infty} p_n = P_n$ and $\lim_{n \to \infty} \sum_{k=1}^{n} n_k = \sum_{n=1}^{\infty} n_n = N_n$ where $P_n < \infty$ and $N_n < \infty$. This implies that

$$\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} p_n - \sum_{n=1}^{\infty} n_n = P_n - N_n$$

so that the series $\sum_{n=1}^{\infty} c_n$ also converges, and we are done.

This result is capable of handling the convergence of many series; however, it is not sufficient for all convergent series. For example, consider the alternating harmonic series

$$\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

The series of absolute values is given by

$$\sum_{n=1}^{\infty} |c_n| = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

which is not convergent (it is the harmonic series, or $p$-series with $p = 1$)—nevertheless, the alternative harmonic series converges! Series which converge despite not being absolutely convergent are called conditionally convergent.

One method of searching for conditionally convergent series is the following:

**Proposition 1.2** (Alternating Series Test). An alternating series $\sum_{n=1}^{\infty} c_n$ converges if the sequence of absolute values of the terms $\{|c_n|\}$ is decreasing toward zero.

**Justification:** We consider the sequence of partial sums $\{S_n\}$. We recall from our study of oscillating sequences that if a sequence alternatively jumped up and then down, and the absolute difference between successive terms decreased to zero, then the sequence converged. We have $S_n - S_{n-1} = c_n$—since we know $c_n$ alternates in sign, so does $S_n - S_{n-1}$! Furthermore, we have $|S_n - S_{n-1}| = |c_n|$ is decreasing to zero so that our sequence of partial sums must converge by the oscillating sequence test. This is just another way of saying that our series converges, and we are done.
Example 1:

Reconsider the alternating harmonic series

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} . \]

Clearly, the sequence is alternating and the terms \( \{|c_n|\} = \{1/n\} \) decrease to zero. By the alternating series test, the series converges.

Example 2:

Determine whether the series

\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2} + 1} \]

converges or diverges.

This is an alternating series so we consider the series of absolute values given by

\[ \sum_{n=1}^{\infty} |c_n| = \sum_{n=1}^{\infty} \frac{n}{n^3 + 1}. \]

For large \( n \) the terms of the series behave like \( 1/n^2 \). By the limit comparison test we have

\[
\lim_{n \to \infty} \frac{n}{n^3 + 1} = \lim_{n \to \infty} \frac{n}{n^3} \cdot \frac{1}{1 + \frac{1}{n^3}} = 1.
\]

Since this converges to a finite number, we have that the original series shares the convergence properties of the function we compared it to. Since \( 1/n^2 \) converges (\( p \)-series with \( p = 2 \)), we have that the original series converges as well.

Example 3:
Determine whether the series
\[ \sum_{n=2}^{\infty} (-1)^{n-1} \frac{\ln(n)}{n} \]
converges or diverges.

We consider the series of absolute values
\[ \sum_{n=2}^{\infty} |c_n| = \sum_{n=2}^{\infty} \frac{\ln(n)}{n}. \]
We know that this series diverges by the comparison test since \(\ln(n)/n > 1/n\) and \(\sum_{n=2}^{\infty} 1/n\) is a divergent series. We may be tempted to conclude that the original series diverges as well—this is not justified. The absolute convergence test is a test for convergence only! It cannot (and this case, does not) imply anything about divergence.

In fact, what we need is the alternating series test. We see that the series alternates because of the \((-1)^{n-1}\) term. Furthermore, we have
\[ \lim_{n \to \infty} \frac{\ln(n)}{n} = \lim_{n \to \infty} \frac{1}{n} = \lim_{n \to \infty} \frac{1}{n} = 0 \]
by l’Hopital’s rule. By the alternating series test, we have that the series converges conditionally.

Example 4:

Determine whether the series
\[ \sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n^2 + 3}}{n^2 + 5} \]
converges or diverges.

We consider the series of absolute values
\[ \sum_{n=1}^{\infty} \frac{\sqrt{n^2 + 3}}{n^2 + 5}. \]
For large \(n\) this behaves like \(1/n\) so we try the limit comparison test to get
\[ \lim_{n \to \infty} \frac{\sqrt{n^2 + 3}}{n^2 + 5} \cdot n = \lim_{n \to \infty} \frac{\sqrt{1 + \frac{5}{n^2}}}{1 + \frac{5}{n^2}} = 1. \]
This implies that series shares the same convergence properties as $\sum_{n=1}^{\infty} 1/n$, which diverges. The series does not converge absolutely.

We now consider the alternative series test. We have that the series alternates because of the $(-1)^n$ term. The absolute values of the term obey

$$\lim_{n \to \infty} |c_n| = \lim_{n \to \infty} \frac{\sqrt{n^2 + 3}}{n^2 + 5} = \lim_{n \to \infty} \frac{\sqrt{n^2 + 3}}{n^2 + 5} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{n \to \infty} \frac{\sqrt{\frac{1}{n^2} + \frac{3}{n^2}}}{1 + \frac{5}{n^2}} = 0.$$

By the alternative series test, we have that the series converges conditionally.