1 Applications of Taylor Series

We have seen how Taylor series can provide a useful approximation of a function on an interval. Furthermore, we have been able to use Taylor’s remainder theorem to bound the maximum error between a function and its Taylor polynomial on an interval.

In fact, there are many further applications of Taylor series—it is one of the most useful tools in applied mathematics. The reason is simple: performing many standard mathematical operations (e.g. differentiation, integration, taking limits, etc.) is difficult for many standard functions; however, it is very easy to perform these operations to monomial terms. Since a Taylor Series consists only of terms of this sort, we can often obtain a very good approximation to the desired solution even when an exact solution to a problem eludes us. Sometimes we can even recover the exact solution itself!

1.1 Limits

We can often use Taylor series to simplify improper limits, limits of the form “0/0” or “∞/∞”. Previously, we have used l’Hopital’s rule—now we will make use the fact that in the limits $x \to 0$ and $x \to \infty$ the monomial terms in the Taylor series have well defined behaviours.

**Example 1:**

Evaluate the limit

$$\lim_{x \to 0} \frac{\sin(x)}{x}.$$  

This is a limit of the form “0/0”. Previously, we had evaluated this limit using l’Hopital’s rule; however, since we now know the Taylor series expansion of $\sin(x)$ we can use this to simplify the limit.
\[
\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}{x} = \lim_{x \to 0} \left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right] = 1.
\]

We recall that by l’Hopital’s rule we had obtained
\[
\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{\cos(x)}{1} = 1.
\]

**Example 2:**

Evaluate the limit
\[
\lim_{x \to 0} \frac{x}{e^x - 1}.
\]

This is a limit of the form "0/0". We use the Taylor series expansion of \(e^x\) to obtain
\[
\lim_{x \to 0} \frac{x}{e^x - 1} = \lim_{x \to 0} \frac{x}{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots - 1} = \lim_{x \to 0} \frac{x}{x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots} = \lim_{x \to 0} \frac{1}{1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots} = 1.
\]

Using l’Hopital’s rule, we had obtained
\[
\lim_{x \to 0} \frac{x}{e^x - 1} = \lim_{x \to 0} \frac{1}{e^x} = 1.
\]

**Example 3:**

Evaluate the limit
\[
\lim_{x \to \infty} xe^{-x}.
\]

We have a limit of the form "\(\infty \cdot 0\)". We can rearrange and use the Taylor expansion for \(e^x\) to get
\[
\lim_{x \to \infty} xe^{-x} = \lim_{x \to \infty} \frac{x}{e^x} = \lim_{x \to \infty} \frac{x}{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots} = \lim_{x \to \infty} \frac{1}{\frac{1}{x} + 1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots}.
\]
We know that 1/x tends to zero as x tends to infinity, and the rest of the terms on the denominator explode as x gets large. This implies

$$\lim_{x \to \infty} xe^{-x} = 0.$$ 

Using l’Hopital’s rule, we had obtained

$$\lim_{x \to \infty} xe^{-x} = \lim_{x \to \infty} \frac{x}{e^x} = \lim_{x \to \infty} \frac{1}{e^x} = 0.$$ 

1.2 Evaluation of Series

Sometimes we are able to use power series to determine the exact value of series. For example, consider the series

$$\sum_{n=0}^{\infty} \frac{1}{n!}.$$ 

We can easily determine by the ratio test that this series converges since

$$L = \lim_{n \to \infty} \left| \frac{c_{n+1}}{c_n} \right| = \lim_{n \to \infty} \left| \frac{n!}{(n+1)!} \right| \lim_{n \to \infty} \left| \frac{1}{n+1} \right| = 0.$$ 

This, however, tells us nothing about what the series converges to. Now that we have a firm grasp of power series—and in particular the forms of the Taylor series expansion for the common functions $e^x$, $\sin(x)$, $1/(1-x)$, etc.—we recognize that this series is just the power series $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ evaluated at $x = 1$. We recognize this series as the Maclaurin series expansion of $e^x$ so that we have

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \quad \text{evaluated at } x = 1 \Rightarrow \sum_{n=0}^{\infty} \frac{1}{n!} = e.$$ 

We can use this association for a wide variety of examples. Often, however, we will have to make use of the tricks (e.g. integration, derivation, etc.) that we have used in previous lectures. We will also have to consider the range on which a given power series converges.
Example 1:

Evaluate the series
\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{2n}}. \]

We start by manipulating so that we have
\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{2n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{4^n} = \sum_{n=1}^{\infty} \left( -\frac{1}{4} \right)^n. \]

This looks a lot like a geometric series, so we consider the power series
\[ \sum_{n=1}^{\infty} x^n = \frac{x}{1-x}. \]

We know that this series converges for \(|x| < 1\), which is satisfied for the point \(x = -1/4\). We have
\[ \sum_{n=1}^{\infty} x^n = \frac{x}{1-x} \quad \overset{x=-1/4}{\Rightarrow} \quad \sum_{n=1}^{\infty} \left( -\frac{1}{4} \right)^n = \frac{-1/4}{1 + 1/4} = \frac{1}{5}. \]

Example 2:

Find the Maclaurin series for \(\arctan(x)\) and use it to evaluate
\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{2n + 1}. \]

We do not want to start by trying to find the Maclaurin series for \(\arctan(x)\) directly. Rather, we notice that
\[ \frac{d}{dx} \arctan(x) = \frac{1}{1 + x^2} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} \]
since \(1/(1 + x^2)\) is a geometric series in \(-x^2\). We now integrate to obtain
\[ \int \frac{d}{dx} \arctan(x) \, dx = \arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n + 1} x^{2n+1} + C. \]
We evaluate at \( x = 0 \) to obtain \( \arctan(0) = C \), which implies \( C = 0 \). We are left with

\[
\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}.
\]

This has a radius of convergence given by

\[
R = \lim_{n \to \infty} \frac{c_n}{c_{n+1}} = \lim_{n \to \infty} \left| \frac{(-1)^n}{2n+1} \cdot \frac{2(n+1) + 1}{(-1)^n} \right| = \lim_{n \to \infty} \frac{2n+3}{2n+1} = 1.
\]

We therefore have the open interval of convergence \(-1 < x < 1\). At the endpoint \( x = -1 \) we have

\[
\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1}
\]

which converges by alternating series test, and the endpoint \( x = 1 \) we have

\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}
\]

which also converges by the alternating series test. Consequently, the power series converges on \(-1 \leq x \leq 1\).

We notice that our original series looks an awful lot like our power series for \( \arctan(x) \) evaluated at \( x = 1 \) (which is just barely included in our interval of convergence!). We test this intuition to obtain

\[
\arctan(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}
\]

\[
\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} = \arctan(1) - 1 = \frac{\pi}{4} - 1
\]

### 1.3 Definite Integrals

Earlier in the semester, we used various approximation schemes (rectangular rule, trapezoidal rule, and Simpson’s rule) to estimate the value of a definite
integral on some domain. Now consider applying Taylor’s remainder theorem to the integral of a function for which we know the Taylor expansion:

\[ \int_a^b f(x) \, dx = \int_a^b [P_n(x) + R_n(x)] \, dx = \int_a^b P_n(x) \, dx + \int_a^b R_n(x) \, dx. \]

Since \( P_n(x) \) contains only monomial terms, \( \int_a^b P_n(x) \, dx \) can be easily integrated. This is our approximation of the integral on the region.

If the term \( \int_a^b R_n(x) \, dx \) can be bounded, we can obtain an estimate of how closely \( \int_a^b P_n(x) \, dx \) approximates \( \int_a^b f(x) \, dx \). This is our error term.

**Example 1:**

Use the second-order Taylor polynomial for \( \ln(x) \) centred at \( x = 1 \) to estimate

\[ \int_{3/2}^1 \ln(x) \, dx. \]

Determine a reasonable bound for the error.

We estimate \( \int_{1}^{3/2} \ln(x) \, dx \) by replacing it with the second-order Taylor polynomial

\[ P_2(x) = (x - 1) - \frac{(x - 1)^2}{2}. \]

We have

\[
\int_{1}^{3/2} \ln(x) \, dx \approx \int_{1}^{3/2} \left( (x - 1) - \frac{(x - 1)^2}{2} \right) \, dx \\
= \left[ \frac{(x - 1)^2}{2} - \frac{(x - 1)^3}{6} \right]_{1}^{3/2} \\
= \frac{(1/2)^2}{2} - \frac{(1/2)^3}{6} \\
= \frac{1}{8} - \frac{1}{48} = \frac{5}{48} \approx 0.10417.
\]

To estimate the error, we use

\[ R_2(x) = \left. \frac{d^3}{dx^3} \ln(x) \right|_{x = 3/2} \frac{(x - 1)^3}{3!}. \]
On the interval $1 < z_2 < x < 3/2$ we have

$$\left| \int_1^{3/2} R_2(x) \, dx \right| = \left| - \int_1^{3/2} \frac{(x - 1)^3}{3z_2^2} \, dx \right| < \int_1^{3/2} \left| \frac{1}{3z_2^2} \right| |x - 1|^3 \, dx \leq \frac{1}{3} \int_1^{3/2} |x - 1|^3 \, dx = \frac{1}{3} \left[ \frac{(x - 1)^4}{4} \right]_1^{3/2} = \frac{1}{12} \left( \frac{1}{2} \right)^4 = \frac{1}{192} \approx 0.00521.$$ 

So even using only two terms of a Taylor polynomial, we obtain a bound of

$$\int_1^{3/2} P_2(x) \, dx - \int_1^{3/2} R_2(x) \, dx < \int_1^{3/2} \ln(x) \, dx < \int_1^{3/2} P_2(x) \, dx + \int_1^{3/2} R_2(x) \, dx$$

$$\implies \quad 0.10417 - 0.00521 < \int_1^{3/2} \ln(x) \, dx < 0.10417 + 0.00521$$

$$\implies \quad 0.0989 < \int_1^{3/2} \ln(x) \, dx < 0.1094.$$

**Example 2:**

Determine the number of terms required to obtain an estimate of the following integral with an error no greater than $10^{-4}$:

$$\int_0^{\pi/2} \sin(x^2) \, dx.$$

We need to bound the error so that $|\int_0^{\pi/2} R_n(x) \, dx| < 10^{-4}$. We have

$$R_n(x) = \frac{d^{n+1}}{dx^{n+1}} \sin(x) \bigg|_{x=z_n} \frac{(x^2)^{n+1}}{(n+1)!}.$$

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Since all derivatives of \( \sin(x) \) are bounded by \(-1\) and 1 we have that

\[
\left| \int_0^{\pi/2} R_n(x) \, dx \right| \leq \int_0^{\pi/2} \frac{x^{2n+2}}{(n+1)!} \, dx
\]

\[
= \left[ \frac{x^{2n+3}}{(2n+3)(n+1)!} \right]_0^{\pi/2}
\]

\[
= \frac{(\pi/2)^{2n+3}}{(2n+3)(n+1)!}.
\]

We cannot explicitly determine when this is less than the bound \(10^{-4}\), but we can test it for several values of \(n\). We find that the first such value is \(n = 10\). This corresponds to a term of order \((x^2)^{10} = x^{20}\)—i.e., we will not need any terms of order \(x^{21}\) or higher. Since the Taylor expansion for \(\sin(x^2)\) contains terms of orders \(x^2, x^6, x^{10}, x^{14}, x^{18}\), we can see that we are going to need five terms to obtain the necessary bound.