Math 319, Fall 2013, Term Test II
Techniques in Ordinary Differential Equations

Date: Friday, November 15
Lecture Section: 001

Name (printed):

UW Student ID Number:

Discussion Section: (circle)

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Instructions

1. Fill out this cover page completely and make sure to circle your discussion section.
2. Answer questions in the space provided, using backs of pages for overflow and rough work.
3. Show all the work required to obtain your answers.
4. No calculators are permitted.

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1. **Short Answer:**

(a) Consider the mechanical model \( mx''(t) + cx'(t) + kx(t) = 0 \). State a relationship on the constants \( m, c, \) and \( k \) such that the mechanism is critically damped.

\[
mr^2 + cr + k = 0 \implies r = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} \]

(b) Determine the Wronskian of \( y_1(x) = \ln(x) \) and \( y_2(x) = 1/x^2 \). Can \( y_1(x) \) and \( y_2(x) \) constitute a fundamental solution set for a homogeneous second-order differential equation (for \( x > 0 \))?

\[
W(y_1, y_2)(x) = y_1'y_2 - y_1y_2
\]

\[
= \ln(x)\left(-\frac{2}{x^3}\right) - \frac{1}{x}\left(\frac{1}{x^2}\right)
\]

\[
= -\frac{1}{x^2}(2\ln(x) - 1) \neq 0 \quad \text{for} \quad x > 0
\]

\( \implies \) could be fundamental solution set!

2. **True/False:**

(a) The complementary solution of \( y''(x) - 8y'(x) + 20y(x) = e^{4x} \sin(2x) \) is \( y_c(x) = C_1 e^{4x} \sin(2x) + C_2 e^{4x} \cos(2x) \). **[True] / [False]**

\[
r^2 - 8r + 20 = 0 \implies r = \frac{8 \pm \sqrt{64 - 80}}{2} = 4 \pm 2i
\]

(b) The amplitude (i.e. maximal value) of the particular solution of \( y''(x) + 2y'(x) + 5y(x) = \cos(\omega x) \) does not depend on \( \omega \). **[True] / [False]**

Resonance!

(c) Every point \( x_0 \in \mathbb{R} \) is an ordinary point of \( (x^2 + 1)y''(x) + xy'(x) - (x + 1)y(x) = 0 \). **[True] / [False]**

\( R(x) = x^2 + 1 \neq 0 \) \( \checkmark \)
3. Second-Order Differential Equations

(a) Consider the following differential equation:

\[ y''(x) + 4y'(x) + 5y(x) = g(x). \]

Determine the complementary solution \( y_c(x) \) then set-up, but do not attempt to evaluate, the trial form \( y_p(x) \) for the given choices of \( g(x) \) below.

\[
\Rightarrow \quad r^2 + 4r + 5 = 0 \quad \Rightarrow \quad r = -2 \pm 3i
\]

\[ y_c(x) = C_1 e^{-2x} \cos 3x + C_2 e^{-2x} \sin 3x \]

(i) \( g(x) = x^2 e^{-2x} \)

\[ y_p(x) = Ax^2 e^{-2x} + \beta x e^{-2x} + C e^{-2x} \]

(ii) \( g(x) = e^{-2x} \cos(x) \)

\[ y_p(x) = A x e^{-2x} \cos(x) + \beta x e^{-2x} \sin(x) \]

(b) Given that \( y_1(x) = 1/x \) and \( y_2(x) = 1/x^5 \) are fundamental solutions of \( x^2 y''(x) + 7xy'(x) + 5y(x) = 0 \), determine the particular solution of

\[ x^2 y''(x) + 7xy'(x) + 5y(x) = x. \]

[Hint: Remember the standard form for when applying variation of parameters!]

\[ W(y_1, y_2)(x) = y_1 y_2' - y_1' y_2 = \left( \frac{1}{x} \right) \left( \frac{-5}{x^6} \right) - \left( \frac{-1}{x^2} \right) \left( \frac{1}{x^5} \right) = -\frac{5}{x^7} + \frac{1}{x^7} - \frac{1}{x^7} \]

Note: \( y(x) = \frac{x}{x^2} = \frac{1}{x} \)

\[ u_1(x) = \int \frac{y_1(x) g(x)}{W(y_1, y_2)(x)} \, dx = \int \frac{(\frac{1}{x})(x)}{-\frac{5}{x^7}} \, dx = \frac{1}{4} \int x \, dx = \frac{1}{8} x^2 \]

\[ u_2(x) = \int \frac{y_1(x) g(x)}{W(y_1, y_2)(x)} \, dx = \int \frac{(\frac{1}{x})(x)}{-\frac{5}{x^7}} \, dx = -\frac{1}{24} x^6 \]

\[ y_p = u_1(x) y_1(x) + u_2(x) y_2(x) = \left( \frac{1}{8} x^2 \right) \left( \frac{1}{x} \right) + \left( -\frac{1}{24} x^6 \right) \left( \frac{1}{x^5} \right) = \frac{1}{8} x - \frac{1}{24} x = \frac{1}{12} x \]
4. Power Series Solutions

[2] (a) Determine the radius of convergence of the power series:

\[ \sum_{n=2}^{\infty} \frac{(-1)^n3^n}{(2n)!}(x-1)^n \]

\[ \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \left| \frac{(-1)^{n+1}3^{n+1}(2n)!}{(2n+2)(2n+1)} \right| = \lim_{n \to \infty} \frac{3}{(2n+2)(2n+1)} = 0 \]

\[ \Rightarrow \text{Converges everywhere!} \]

[2] (b) Write the following expression as a single summation whose common term is \(x^n\):

\[ \sum_{n=2}^{\infty} (n-1)n^2a_nx^{n-2} + x \sum_{n=0}^{\infty} na_nx^{n-1} \]

\[ = \sum_{n=0}^{\infty} \left[ (n+1)(n+2)^2a_{n+2}x^n + \sum_{n=0}^{\infty} na_nx^n \right] \]

\[ = \sum_{n=0}^{\infty} \left[ (n+1)(n+2)^2a_{n+2} + na_n \right]x^n \]

[2] (c) Suppose that a particular differential equation has a power series solution centered at \(x_0 = 0\) generated by the recursion relation

\[ a_{n+2} = \frac{a_{n+1} - (n+1)a_n}{n+2}, \quad n \geq 0. \]

Determine up to the \(x^3\) term of the solution \(y(x) = \sum_{n=0}^{\infty} a_nx^n\).

\[ n = 0 \quad \Rightarrow \quad a_2 = \frac{a_1 - a_0}{2} \]

\[ n = 1 \quad \Rightarrow \quad a_3 = a_2 - 2a_1 = \frac{(a_1 - a_0) - 2a_1}{3} = -\frac{1}{6}a_0 - \frac{1}{2}a_1 \]

\[ \Rightarrow y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots \]

\[ = a_0 + a_1x + \left( -\frac{a_0}{2} + \frac{a_1}{2} \right)x^2 + \left( -\frac{a_0}{6} - \frac{a_1}{2} \right)x^3 + \ldots \]

\[ = a_0 \left( 1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \ldots \right) + a_1 \left( x + \frac{1}{2}x^2 - \frac{1}{2}x^3 + \ldots \right) \]
5. Laplace Transforms:

(a) Evaluate the following inverse Laplace transform:

\[
\mathcal{L}^{-1}\left\{\frac{2}{s^2 - 4s + 8}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{(s-2)^2 + 4}\right\} = \frac{e^{2x}}{2} \sin(2x)
\]

(b) Use the Laplace transform method to evaluate the following initial value problem:

\[
\frac{d^2 y}{dx^2} - y(x) = e^x, \quad y(0) = 0, \quad y'(0) = 1/2.
\]

\[
\text{Laplace Transform} \Rightarrow \left[ s^2 Y(s) - sy(0) - y'(0) \right] - Y(s) = \frac{1}{s-1}
\]

\[
\Rightarrow (s^2 - 1) Y(s) = \frac{1}{s-1} + \frac{1}{2} = \frac{2 + s - 1}{2(s-1)} = \frac{s+1}{2(s-1)}
\]

\[
\Rightarrow Y(s) = \frac{s+1}{2(s-1)(s^2-1)} = \frac{s+1}{2(s-1)(s-1)(s+1)} = \frac{1}{2(s-1)^2}
\]

\[
\Rightarrow y(x) = \frac{1}{2} x e^x
\]