Instructions

1. Fill out this cover page **completely** and make sure to circle your discussion section.

2. Answer questions in the space provided, using the last page for overflow.

3. Show all the work required to obtain your answers.

4. No calculators are permitted.

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Discussion Section: (circle)

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- **Huanyu Wen:** 305 306 323 324
- **Dongfei Pei:** 325 326 329
- **Kai Hsu:** 327 328
1. **Short Answer:**

(a) Suppose that $y_1(x)$ and $y_2(x)$ are two solutions of the homogeneous second-order differential equation

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0$$

satisfying $W(y_1, y_2)(x) \neq 0$. State the general solution form of the differential equation.

2. **True/False:**

(a) The mechanical mechanism $x''(t) + cx'(t) + 4x(t) = 0$ is critically damped at the value $c = 4$. [True / False]

(b) The amplitude (i.e. maximal value) of $y(x) = 3 \cos(x) - 4 \sin(x)$ is 5. [True / False]

(c) Every point $x_0 \in \mathbb{R}$ is an ordinary point of $(x^2 - 1)y''(x) + xy'(x) - (x + 1)y(x) = 0$. [True / False]
3. Second-Order Differential Equations

(a) Consider the following differential equation:

\[ y''(x) + 6y'(x) + 5y(x) = g(x). \]

Determine the complementary solution \( y_c(x) \) then set-up, but do not attempt to evaluate, the trial form \( y_p(x) \) for the given choices of \( g(x) \) below.

(i) \( g(x) = e^{-5x} \sin(x) \)

(ii) \( g(x) = xe^{-x} \)

(b) Given that \( y_1(x) = 1 + x \) and \( y_2(x) = e^x \) are fundamental solutions of \( xy''(x) - (1 + x)y'(x) + y(x) = 0 \), determine the particular solution \( y_p(x) \) of

\[ xy''(x) - (1 + x)y'(x) + y(x) = 2x^2e^x. \]

[Hint: Remember the standard form when applying variation of parameters!]
4. Power Series Solutions

(a) Determine the radius of convergence of the power series:

\[ \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(n+1)}{n!} x^n \]

(b) Write the following expression as a single summation whose common term is \( x^n \):

\[ x \sum_{n=1}^{\infty} n(n+1)a_n x^{n-2} + \sum_{n=2}^{\infty} (n-1)a_n x^{n-2} \]

(c) Suppose that a particular differential equation has a power series solution centered at \( x_0 = 0 \) generated by the recursion relation

\[ a_{n+2} = \frac{a_{n+1} + a_n}{(n+2)(n+1)}, \quad n \geq 0. \]

Determine up to the \( x^3 \) term of the solution \( y(x) = \sum_{n=0}^{\infty} a_n x^n \).
5. Laplace Transforms:

(a) Evaluate the following inverse Laplace transform:

\[ L^{-1}\left\{ \frac{s + 2}{(s - 1)^2 + 9} \right\} \]

(b) Use the Laplace transform method to solve the following initial value problem:

\[ 2y''(x) - 3y'(x) + y(x) = 0, \quad y(0) = 0, \quad y'(0) = 1/2. \]