Submission instructions: Clearly write your full name and ID number on the first page. To avoid marking and handling difficulties, please staple all submitted pages together and answer the questions in the order they appear on the assignment.

Academic integrity: Students are encouraged to collaborate on assignment problems but must write up their assignments independently. Copying is strictly forbidden!

Problems for submission:

1. Section 7.1: 8

2. Section 7.2: 2, 6(a), 12

3. Section 7.3: 1

4. Use the function $L(x, y) = x^6 + \alpha y^2$ for some $\alpha > 0$ to be determined to prove that $(0, 0)$ is an asymptotically stable fixed point of

$$\begin{cases} 
\frac{dx}{dt} = -x^3 - y \\
\frac{dy}{dt} = x^5 
\end{cases}$$

(Hint: If you use LaSalle’s Invariance Principle, be careful to justify all of the requirements.)

5. Consider a biochemical reaction system involving two substrates, an inactive protein $X$ and an active protein $Y$, subject to the following interactions:

   (i) There is inflow and outflow of the inactive protein $X$ (i.e. we have reactions of the form $\emptyset \rightleftharpoons X$)
(ii) The active proteins reversibly facilitate activation of the inactive proteins (i.e. we have reactions of the form \( X + Y \rightleftharpoons 2Y \)).

After simplification, this gives rise to the \textit{mass action model}

\[
\begin{align*}
\frac{dx}{dt} &= \alpha - x - xy + y^2 \\
\frac{dy}{dt} &= xy - y^2.
\end{align*}
\]

where \( \alpha > 0 \) is a pooled inflow parameter.

(a) Determine all of the fixed points in the region \( \mathbb{R}^2_+ = \{(x, y) \mid x \geq 0, y \geq 0\} \). \textbf{(Hint:} Do not forget to look at the boundary!\textbf{)}

(b) Show that the following function is a Lyapunov function for the positive fixed point found in part (a), and that it is decreasing along solutions of (1):

\[
L(x, y) = x(\ln(x) - \ln(\alpha) - 1) + y(\ln(y) - \ln(\alpha) - 1) + 2\alpha.
\]

What can we conclude based on Lyapunov function theory? What does this mean for the original physical system? \textbf{(Hint:} As in class, to show that \( L'(t) < 0 \) you will have to use the inequality \((\alpha - \beta)(\ln(\beta) - \ln(\alpha)) \leq 0\) where equality is attained if and only if \( \alpha = \beta \).\textbf{)}

6. Consider a prey \((x)\) and predator \((y)\) model with the following interactions:

(i) Linear growth in the prey (i.e. proportional to \( x \)) and linear decay in the predator (i.e. proportional to \( y \))

(ii) Quadratic decay in both species (i.e. proportional to \( x^2 \) and \( y^2 \))

(iii) An interaction term which hurts the prey and helps the predator (i.e. proportional to \( xy \)).

After rescaling some parameters, we can capture these basic interactions with the model

\[
\begin{align*}
\frac{dx}{dt} &= x(\beta - x - \alpha y) \\
\frac{dy}{dt} &= y(-1 + \alpha x - y)
\end{align*}
\]

where \( \alpha > 0 \) is a pooled interaction term and \( \beta > 0 \) is a pooled growth term for the prey.
(a) Determine all fixed points in the region $\mathbb{R}^2_{\geq 0} = \{(x, y) \mid x \geq 0, y \geq 0\}$. What are the conditions (if any) on the parameters $\alpha > 0$ and $\beta > 0$ in order for there to be a strictly positive fixed point? Interpret this condition in terms of the physical system.

(b) Determine the nullclines of the system (2) for the parameter values $\alpha = 2$ and $\beta = 1$ and use linearization to classify the positive fixed point $(\bar{x}, \bar{y}) = (3/5, 1/5)$. Use this information to sketch the vector field diagram in the region $\mathbb{R}^2_{\geq 0} = \{(x, y) \mid x \geq 0, y \geq 0\}$.

(c) Show that the function

$$L(x, y) = -3 \ln(x) - \ln(y) + 5(x + y) + 3 \ln(3/5) + \ln(1/5) - 4$$

satisfies the criteria to be a Lyapunov function of the system (2) for $\alpha = 2$ and $\beta = 1$.

(d) Show that $\frac{dL}{dt} < 0$ for all $(x, y) \in \mathbb{R}^2_{\geq 0}$ except $(3/5, 1/5)$. What can we conclude about trajectories of the system (2) starting in the region $\mathbb{R}^2_{> 0} = \{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$? Interpret this result in terms of the physical system.

**Bonus! (Warning: very messy!)** Show that, whenever a strictly positive fixed point $(\bar{x}, \bar{y})$ of (2) exists, the function

$$L(x, y) = -\left(\alpha + \beta\right) \ln(x) - \left(\alpha \beta - 1\right) \ln(y) + (\alpha^2 + 1)(x + y) + C$$

is a Lyapunov function relative to it. Determine the value of $C$ and show that

$$\frac{dL}{dt} < 0$$

along solutions not originating at the fixed point.