Instructions

1. Fill out this cover page.
2. Answer questions in the space provided, using back page for overflow and rough work.
3. Show all work required to obtain your answers.
4. Unless otherwise stated, you may use any theorem or result derived in class.

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1. Definitions and Theorems:

Consider a first-order autonomous system of differential equations:

\[
\begin{align*}
\frac{dx}{dt} &= f_1(x, y) \\
\frac{dy}{dt} &= f_2(x, y).
\end{align*}
\]  

(1)

(a) State what it means for a fixed point \( \bar{x} = (\bar{x}, \bar{y}) \) of (1) to be Lyapunov stable.

\[ \forall \varepsilon > 0, \exists \delta > 0 \text{ such that} \]
\[ ||\bar{x}(0) - \bar{x}|| < \delta \implies ||\bar{x}(t) - \bar{x}|| < \varepsilon \quad \forall t \geq 0. \]

(b) State the Poincaré-Bendixson Theorem.

Consider a planar (i.e. 2D) region \( R \) which is closed and bounded and has no fixed points. If solutions may not leave \( R \), then there is a periodic orbit in \( R \).

(c) Give an example of a non-trivial system (1) which is Hamiltonian and give the corresponding Hamiltonian function \( H(x, y) \).

\[ H(x, y) = x + y \]

\[ \Rightarrow \quad \begin{align*}
x' &= 1 \\
y' &= -1
\end{align*} \]

(d) Suppose that (1) has the solutions pictured below. Sketch a vector field which could plausibly generate the given solution. Overlay the given solution on this picture.
2. Vector Fields and Linearization:

Consider the following system of differential equations:

\[
\begin{align*}
\frac{dx}{dt} &= x^2 + y^2 - 2 \\
\frac{dy}{dt} &= x^2 - y
\end{align*}
\]

(a) Sketch the vector field and identify the nullclines.

1. \( x' = 0 \) \( \Rightarrow \) \( x^2 + y^2 = 2 \) \( \Rightarrow \) circle of radius \( \sqrt{2} \)
2. \( y' = 0 \) \( \Rightarrow \) \( y = x^2 \)
3. \( y' > 0 \) \( \Rightarrow \) parabola
4. \( y' < 0 \) \( \Rightarrow \) parabola

(b) Determine all fixed points and, if possible, their linear stabilities.

1. \( y' = 0 \) \( \Rightarrow \) \( y = x^2 \)
2. \( x' = 0 \) \( \Rightarrow \) \( x^2 + y^2 = 2 \) \( \Rightarrow \) \( y^2 - z = 0 \)
   \( \Rightarrow \) \( y = \pm 1 \)

For \( (0, 0) \), the Jacobian is:

\[
Df(x, y) = \begin{bmatrix} 2x & 2y \\ 2x & -1 \end{bmatrix}
\]

\( Df(0, 0) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow (2-x)(-1-x)-4 = x^2 - x - 6 = (x-3)(x+2) = 0 \)

\( \Rightarrow x = 3, x = -2 \)

\( \Rightarrow \text{saddle} \)

For \( (-1, 0) \), the Jacobian is:

\[
Df(x, y) = \begin{bmatrix} -2 & 2 \\ -2 & -1 \end{bmatrix}
\]

\( Df(-1, 0) = \begin{bmatrix} -2 & 2 \\ -2 & -1 \end{bmatrix} \Rightarrow (-2-x)(-1-x)+4 = x^2 + 3x + 6 = 0 \)

\( \Rightarrow x = -\frac{3 \pm \sqrt{9-24}}{2} = -\frac{3 \pm 5i}{2} \)

\( \Rightarrow \text{spiral sink} \).
3. Conservative Systems:

(a) Show that the following system is Hamiltonian and determine the Hamiltonian function \( H(x, y) \):

\[
\begin{align*}
\frac{dx}{dt} &= \frac{y}{1 + x^2} \\
\frac{dy}{dt} &= x + xy^2 \\
\frac{dH}{dt} &= 0
\end{align*}
\]

\[
\frac{df}{dx} + \frac{df}{dy} = -\frac{2xy}{(1+x^2)^2} + \frac{2xy}{(1+x^2)^2} = 0, \quad \Rightarrow \text{Hamiltonian,}
\]

\[
\frac{\partial H}{\partial y} = \frac{y}{1+x^2} \quad \Rightarrow \quad H(x, y) = \frac{y^2}{2(1+x^2)} \Rightarrow \frac{\partial H}{\partial x} = -\frac{xy^2}{(1+x^2)^2} + y^1(x)
\]

\[
\frac{\partial H}{\partial x} = -\frac{x}{(1+x^2)^2} \quad \Rightarrow \quad \theta^1(x) = \frac{-x}{(1+x^2)^2}
\]

\[
\Rightarrow \quad \theta^0(x) = -\frac{1}{2(1+x^2)}
\]

\[
\Rightarrow \quad H(x, y) = \frac{y^2 + 1}{2(1+x^2)}
\]

[b] Consider the two-dimensional autonomous system \( x' = f_1(x, y), y' = f_2(x, y) \). Suppose that there is a function \( H(x, y) \) such that

\[
\frac{dy}{dx} = -\left( \frac{\partial H}{\partial x} \right) \left/ \left( \frac{\partial H}{\partial y} \right) \right.
\]

Prove that the system is conservative. [Hint: Note that \( y \) depends upon \( x \) which in turn depends upon \( t \).]
4. Application I: Mass-spring model

Consider a mass-spring system which experiences a linear restoring force and a nonlinear damping force. Using Newton's second law, we will assume the model is given by

\[ m x'' = -c x^3(x') - k x \]  

(2)

where \( m > 0 \) is the mass of the system, \( c > 0 \) is the damping parameter, and \( k > 0 \) is the restoring parameter.

(a) Explain the physical effect of the nonlinear damping term \(-c x^3(x')\) as compared to the standard linear damping term \(-c x'\).

The system is more damped far away from \( x = 0 \) and less damped close to \( x = 0 \).

(b) Sketch the vector field diagram of (2) in the \((x, x')\)-plane (or the \((x_1, x_2)\)-plane).  
[Hint: Note that picture does not depend upon the parameters, so you may choose \( m = c = k = 1 \).]

\[
\begin{align*}
x_1 &= x \\
x_2 &= x' \\
x_1' &= -c x_1^2 x_2 - \frac{k}{m} x_1 \\
x_2' &= -\frac{c}{m} x_1^2 x_2 - \frac{k}{m} x_1
\end{align*}
\]

(c) Show that the total energy function \( E(x, x') = \frac{m}{2} (x')^2 + \frac{k}{2} x^2 \) is a nonincreasing Lyapunov function of the fixed point \((x, x') = (0, 0)\). Use a result from class to prove that \((0, 0)\) is an asymptotically stable fixed point of (2).

\[
\frac{dE}{dt} = \frac{\partial E}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial E}{\partial x_2} \frac{dx_2}{dt} = (k x_1)(x_2) + (m x_1)(-\frac{c}{m} x_1^2 x_2 - \frac{k}{m} x_1)
\]

\[= -c x_1^2 x_2^2 \leq 0.\]

(Note \( x_1 = 0, x_2 \neq 0 \Rightarrow x_1' = x_2 \neq 0 \Rightarrow \) Not invariant.  \( x_1 \neq 0, x_2 = 0 \Rightarrow x_2 ' = -\frac{k}{m} x_1 \neq 0 \) \Rightarrow \) Not invariant.

LaSalle's Invariance Principle \Rightarrow asymptotically stable.)
5. **Application II: Infection model**

Consider the following model of disease spread between a susceptible population ($S$) and an infected population ($I$):

$$\begin{align*}
\frac{dS}{dt} &= S + I - \alpha SI - S^2 \\
\frac{dI}{dt} &= \alpha SI - 2I
\end{align*}$$

where $\alpha > 0$ is a pooled transition rate which roughly measures how likely a susceptible individual is to contract the disease when contact is made with an infected individual.

(a) Determine all fixed points of the system (3) and interpret them in terms of the physical model. State any conditions on $\alpha$ required for the existence of an *endemic* fixed point (i.e., a fixed point with $I > 0$).

$$\begin{align*}
I = 0 & \Rightarrow \ I = (\alpha S - 2) = 0 \Rightarrow 2 = \frac{\alpha}{2} \\
S = 0 & \Rightarrow 0 = 5 - S = 0 \Rightarrow S = 1 \text{ or } S = 0 \Rightarrow (1, 0), (0, 0) \\
S = \frac{2}{\alpha} & \Rightarrow \ S = \frac{2}{\alpha} + I - 2I - \frac{4}{\alpha} = 0 \Rightarrow I = \frac{2}{\alpha}(1 - \frac{3}{2}) = \frac{2}{\alpha}(\frac{1}{2}) = \frac{1}{\alpha} \\
\Rightarrow \left( \frac{2}{\alpha}, \frac{1}{\alpha} \right) & \text{ is positive if } \alpha > 2.
\end{align*}$$

(b) For $\alpha = 1$, sketch the vector field of (3) in the region $\{(S, I) \mid S \geq 0, I \geq 0\}$. Conjecture as to whether the long-term behavior is endemic or non-endemic.

[Hint: Note that the nullcline for $S' = 0$ is given by $I = \frac{S(S - 1)}{1 - \alpha S}$.]

$$\begin{align*}
S' &= S + I + SI - S^2 = 0 \Rightarrow I = \frac{S(S - 1)}{1 - S} = -S \quad \text{but sign changes at } S = 1 \\
I' &= I(S - 2) = 0 \Rightarrow I = 0 \text{ or } S = 2 \\
\Rightarrow S' &= 0 \quad \text{Also, } S = 1 \Rightarrow S' = 0.
\end{align*}$$

$$(0, 1) \text{ is stable.}$$
(c) Use linear stability analysis to classify the fixed points (i) \((1, 0)\) and (ii) \(\left(\frac{\alpha}{\alpha}, \frac{2}{\alpha} \frac{(\alpha - 2)}{\alpha}\right)\).

Note if the stability depends upon \(\alpha\). [Hint: For the second fixed point, note that a fixed point is stable if \(\text{tr}(Df(\vec{s}, \vec{t})) < 0\) and \(\text{det}(Df(\vec{s}, \vec{t})) > 0\).]

\[
Df(\vec{s}, \vec{t}) = \begin{bmatrix}
-\alpha & -2s \\
\alpha s & -2
\end{bmatrix}
\]

\[
Df(1, 0) = \begin{bmatrix}
-1 & 1 - \alpha \\
0 & \alpha - 2
\end{bmatrix} \Rightarrow \lambda_1 = -1, \lambda_2 = \alpha - 2
\]

Sink if \(\alpha < 2\), saddle if \(\alpha > 2\).

\[
Df(\frac{\alpha}{\alpha}, \frac{2}{\alpha} (\frac{\alpha - 2}{\alpha})) = \begin{bmatrix}
1 - \frac{2}{\alpha} (\alpha - 2) - \frac{\alpha}{\alpha} & -1 \\
\frac{2}{\alpha} (\alpha - 2) & 0
\end{bmatrix} = \begin{bmatrix}
-1 & -1 \\
\frac{2}{\alpha} (\alpha - 2) & 0
\end{bmatrix}
\]

\[
\text{tr}(Df) = -1 < 0 \quad \det(Df) = \frac{2}{\alpha} (\alpha - 2) \quad \text{if} \quad \alpha > 2
\]

Source if \(\alpha < 2\)
Sink if \(\alpha > 2\)

(d) Interpret what happens to the physical system as the pooled transmission rate \(\alpha\) passes from the non-endemic region to the endemic region. What type of bifurcation is this and what is the bifurcation value?

Transcritical bifurcation at \(\alpha = 2\).

Non-endemic fixed point \((0, 1)\) (always exists!) loses stability at \(\alpha = 2\) (as \(\alpha > 2\))

Endemic fixed point \((\frac{\alpha}{\alpha}, \frac{2}{\alpha} (\frac{\alpha - 2}{\alpha}))\) passes into endemic region and gains stability.