MATH 415, WEEK 1:
Introduction, Definitions & Review

1 Introduction

This course is intended to be a broad introduction to the study **dynamical systems** and **modeling**.

You will probably find that this course is unlike the mathematics courses you have become used to taking, which have focused exhaustively and narrowly on the study of a specific topic. While dynamical systems as a discipline is well-studied in its own right, the techniques are drawn from an assortment of areas. It is equal parts **calculus, differential equations, algebra, geometry, analysis, computation**, and **natural science**. We will delve at an introductory level into each of these areas; and yet none can be said to encapsulate the discipline. We will be careful to fill in the missing details as we go.

As far as background material goes, I will be assuming a familiarity with the basics of **multivariate calculus** (Math 234 or equivalent), **linear algebra** (Math 320 or 340), and that you have at least one course in **ordinary differential equations** (Math 319 or 320). Familiarity with **mathematical analysis** (Math 521) and **computational methods** (Math 514) are beneficial but not required. I realize some of you may not have had a formal introduction to linear algebra as it is not strictly required (if you took Math 319 with an instructor who did not emphasize it). This should not be a significant problem, but it is advisable to catch up early in the semester if you can. Eigenvalues and eigenvectors factor prominently in the analysis of dynamical systems and the more you understand about them, the smoother the course will go. I will post background material as it becomes relevant.

2 Dynamical Systems

We have that dynamical systems do not fit snuggly within one single classification of mathematics, so what exactly are they? The answer is as trivial as it appears:

Definition 2.1. *A **dynamical system** is given by a set of objects which undergoes change over time according to a fixed rule.*

After a moment's thought, we realize that we already know many things with change of time: the weather, the voltage in a circuit, the value of a stock. All of these things can be imagined as dynamical system—in fact, with the definition given, just about *everything* is a dynamical system!

And this is absolutely true. Dynamical systems are everywhere. The key insight we are about to make is to formalize this intuition mathematically. To do this, we need to be precise. After a little thought, we should agree that what we really need are two things:

- A set of **states** which tells us at each instance of time exactly how the system looks (e.g. is it raining? how is my mutual fund doing? etc.).
- A **transition rule** which tells us how one state connects to the next (e.g. is the wind blowing the storm away? are my funds due for a dive? etc.).

We will consider a number of dynamical systems within this course. While many will appear to have more differences than similarities, the division into states and transitions will be found in all of them.

We will focus on two types of dynamical systems in this course:

1. **Differential equations** ($\sim$12 weeks)

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}(t, \mathbf{x}(t)), \qquad \mathbf{x}(0) = \mathbf{x}_0$$

   where $\mathbf{x} \in \mathbb{R}^m$, $t \in [a, b] \subseteq \mathbb{R}$, and $\mathbf{f} : [a, b] \times \mathbb{R}^m \mapsto \mathbb{R}^m$; and

2. **Difference equations** ($\sim$3 weeks)

$$\mathbf{x}_{n+1} = \mathbf{f}(n, \mathbf{x}_n),$$

   where $\mathbf{x}_n \in \mathbb{R}^m$, $n \in \mathbb{N}$, and $\mathbf{f} : \mathbb{N} \times \mathbb{R}^m \mapsto \mathbb{R}^m$.

The difference between these two classifications is subtle but important. For differential equations the transitions from state-to-state occur **continuously** while for difference equations they occur in **discrete jumps**. We can quickly divide our previous examples into these classifications. We imagine that the weather changes continuously with time while the updating of stock values occurs discretely, for instance.

(It should be noted that these are not the furthest reaches of what is encapsulated by the discipline of dynamical systems. In this course we will imagine the transition rule is **deterministic**. That is, we suppose that,

given information at one state, we can determine without any ambiguity or doubt what the next state must be. This, of course, is not realistic in many applications—consider quantum mechanics. It is often more realistically that each state depends probabilistically on the previous state(s). Such a system is called a **stochastic dynamical system** and is the object study of Math 605 and 632. There is also the related discipline of **partial differential equations**, which are common in spatial problems, but are not covered in this course. See Math 322 for more information.)

3 Our understanding of ODEs so far...

Let's start with the dynamical systems which are most familiar to us: **ordinary differential equations** (ODEs). Suppose we want to model the growth of a colony of bacteria. The challenge posed to us as mathematicians (or applied scientists with an appreciation for the power of mathematics!) is to distill from this problem a mathematical formulation. In this case, we are looking for two things: a *state* and a *transition rule*.

In this case, the state is more or less obvious—we want to track the *population* of the colony over time. To make things simple, we will assign this the variable $P(t)$. We may think of $P(t)$ as giving the *state of the system* at time t. The transition rule is a little more complicated. In fact, there is no single answer to what we should use. Context will play a key role. Let's suppose for a moment that we are not interested in the details of how the bacteria grow and that there are a great many of them. We can imagine, then, that the population $P(t)$ is a continuous variable (a lie, but a relatively insignificant one if the population is large). Suppose we assign the following transition rule:

- The rate of the population's change at time t is proportional to the size of the population at time t.

This seems reasonable in broad strokes. Roughly, if the population is small, there will only be small growth, while if the population is larger, there will be larger growth. This gives rise to the first-order linear differential equation

$$\frac{dP(t)}{dt} = kP(t), \qquad P(0) = P_0$$

where P_0 is the initial population size and $k > 0$ is some proportionality constant. We can quickly solve this DE (it is separable, first-order linear, and also easy to solve by guessing) to get the solution $P(t) = P_0 e^{kt}$. Recall

that, by a **solution** of a differential equations, we mean a function which satisfies the given equation. In this case, we can easily verify that

$$\frac{d}{dt}P(t) = P_0\frac{d}{dt}e^{kt} = k[P_0e^{kt}] = kP(t)$$

and that

$$P(0) = P_0e^{k(0)} = P_0.$$

At this point, we can also quickly answer **quantatative** and **qualitative** questions about the system. For instance, if we know that the initial population is a hundred bacteria (i.e. $P_0 = 100$) and the proportionality rate is $k = 0.5$, we can now state definitively that the population after two years (i.e. $t = 2$) is $P(2) = 100e^{0.5(2)} \approx 272$. In our dynamical systems lingo, we might say that the *state of the system* is $P = 272$ at time $t = 2$. We can also questions about qualitative dynamics. For instance, we can say that the system is unbounded in time (or that it explodes) since

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} P_0e^{kt} = \infty$$

for all $P_0 > 0$. That is to say, the bacteria colony grows without bound. This should caution us that our model may not be entirely reasonable!

In Math 319 and 320 we encountered many examples which were more complicated than this. Consider the problem of tracking the position of a pendulum which swings as a result of gravity. A similar model can be derived by considering a mass/spring system obeying Hooke's law (see Figure 1).

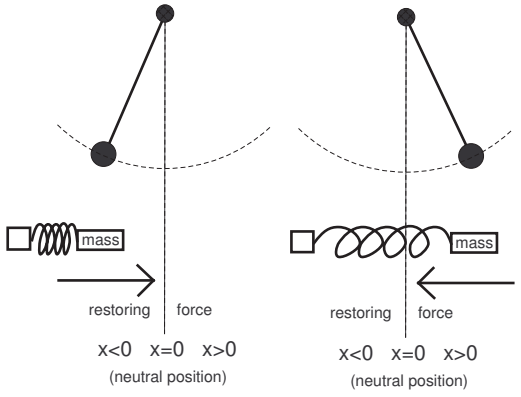

Figure 1: Illustration of the so-called restoring force acting on a swing pendulum or mass/spring obeying Hooke's law.

The state variable we are interested in is the position of the pendulum at time t, which we will denote $x(t)$ and choose to center at $x = 0$ at the center of the pendulum. For the transition rule, we appeal to Newton's second law $F = ma$. Disregarding friction, we imagine that there is a *restoring force* which acts on the pendulum. Roughly, if the pendulum swings to the right, gravity acts to push it left, while if it is swinging left, gravity pushes it to the right. Assuming this force is proportional to the magnitude of displacement, we arrive at the second-order linear differential equation

$$\begin{aligned} & F = ma \\ \implies \quad & -kx(t) = m\frac{d^2x(t)}{dt^2} \\ \implies \quad & \frac{d^2x(t)}{dt^2} + \frac{k}{m}x(t) = 0 \end{aligned}$$

where m is the mass of the pendulum and k is a proportionality constant for the restoring force. We were able to solve DEs of this type in Math 319 and 320 by guessing $x(t) = e^{rt}$ and using some identities. We eventually arrived at $x(t) = C_1 \sin\left(\sqrt{\frac{k}{m}}t\right) + C_2 \cos\left(\sqrt{\frac{k}{m}}t\right)$ where C_1 and C_2 are constants which can be determined by the initial conditions (position and velocity). If we are unconvinced, we can verify that this is a solution by substituting in the DE (check!).

In all, in Math 319 and 320 we were able to solve differential equations of a wide variety of general forms. The most common forms we studied were **separable equations**, **first-order linear equations**, **Bernoulli DEs**, **exact DEs**, **higher-order constant coefficient DEs** (homogeneous of nonhomogeneous), and **first-order linear systems**. We may also be familiar with **Laplace transform** and **power series** methods. We will not need all of these techniques in this course (for reasons we will see shortly) but you should refamiliarized yourself with first-order linear and separable DEs, as well as first-order linear systems, if you have forgot about them.

We should feel fairly confident in our abilities in solving differential equations, but consider the following observation regarding our pendulum example. We assumed that the restoring force was proportional to the displacement which we calculated along a straight line. This is reasonable near the resting position $x = 0$, but certainly not far away. In particular, the pendulum's trajectory will necessarily curve away from $x = 0$ which is not captured in this given model. The straight line model will never be able to capture the phenomenom of the pendulum swinging over the top, for instance.

A more realistic model is to track to the state of the system by the *angle* the pendulum makes with the pivot at time t. We will denoted this by $\theta(t)$. It can be derived that this gives the related model

$$\frac{d^2\theta(t)}{dt} + \frac{g}{L}\sin(\theta(t)) = 0$$

where g is the force due to gravity and L is the length of the pendulum. This model is very similar to the one we have already considered—nevertheless, it can be quickly checked that the methods established in Math 319 and 320 are utterly helpless in determining the solution. In fact, we will see that even attempting to find an explicit solution for this model is a somewhat misguided—the solution is far too convoluted and unwieldy to be of practical use. How can such a small change to the DE make such a large difference in our approach?

The difference is that the state variable $\theta(t)$ appears **nonlinearly** in the equation (in the term $\sin(\theta(t))$). This may seem like a small change but in all but the simplest of cases it changes our approach entirely. In some senses it represents a transition to the big leagues of mathematical analysis. We will no longer have the tools afforded by linearity in searching for solutions (e.g. splitting solution into homogeneous and particular parts). In many cases, the equations will be such that we will have no hope of finding an analytic solution at all. But if we can't find the solution, what is there left for us to do?

In fact, we have been given a false sense of confidence in Math 319 and 320 in that our first step was almost always to look for the analytic solution. For most problems arising from the sciences, this is not a practical approach. But it is not the only approach. Broadly, we have the following options available to us:

1. **Analytic approach:** This was the approach employed in Math 319 and 320, as well as in earlier calculus courses. All of the methods indicated above were specifically designed to find an explicit analytic solution. We will quickly find in this course that it is not a viable option for most real-world problems.

2. **Qualitative approach:** In this approach, we are interested in determining qualitative behavior of solutions without having to find solutions. For instance, we are interested in questions such as whether solutions converge to fixed points, explode, or oscillate. Most of the analysis contained in this course will fall within this the scope of qualitative approaches.

3. **Numerical approach:** In this approach, the system is simulated on a computer using a carefully crafted algorithm. Numerical approaches have gained prominence over the past half century as computers have become more powerful; nevertheless, there are still many insufficiencies in this approach. We will dabble in numerical methods but be careful not to become overly reliant on it.

The difference between the first and second approach is important enough to draw attention to again. **For most of this course, we will not be interested at all in finding the analytic solutions to the differential equations we encounter**. We will treat the question of finding a function $x(t)$ satisfying a given relation as insurmountable. This may be because it is too hard, but it may also be because it is impossible—for some of the systems we will analyze, no analytic solution will exist *even in principle*.

4 Autonomous First-Order Systems of Differential Equations

We are often interested in **systems** with multiple dynamic and inter-dependent variables (e.g. the position of a particle in multiple dimensions, the concentrations of multiple reacting chemical species over time, etc.). One way to represent this is as a **system of autonomous first-order differential equations** of the form

$$\begin{aligned} \frac{dx_1}{dt} &= f_1(x_1, \ldots, x_n) \\ \vdots &\qquad\qquad \vdots \\ \frac{dx_n}{dt} &= f_n(x_1, \ldots, x_n) \end{aligned} \tag{1}$$

where $x_i \in \mathbb{R}$, $i = 1, \ldots, n$, are the n state variables we are tracking. It is convenient to express the system in the condensed vector form

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x})$$

where $\mathbf{x}(t) = (x_1(t), \ldots, x_n(t)) \in \mathbb{R}^n$ and $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_n(\mathbf{x})) \in \mathbb{R}^n$.

There are a few things to notice about this set-up. First of all, the system is **first-order**—that is to say, the highest order derivative of any variable appearing in the expression is a first-order derivative. (We can define second- and third-order DEs and systems of DEs in the same way.)

This is a nice property to have since there is a natural interpretation of first-order derivatives as *rates*. A positive derivative means the variable is increasing while a negative derivative means the variable is decreasing. We will see that this is advantageous when consider **geometric** and **numerical** approaches to differential equations.

The second thing to notice is that the system is **autonomous**. Autonomous differential equations are those which do not depend explicitly on time. (Otherwise, we would have terms $f_i(t, x_1, \ldots, x_n)$ on the right-hand side.) The transition rule from autonomous dynamical system depends only upon the **current state of the system**. Such systems are common in physics, in particular, since the physical laws governing the universe are assumed not to change over time (e.g. gravity, magnetic/electrical attraction). Non-autonomous systems frequently arise due to **external forcing** (e.g. season-dependent population growth models, forced pendulums, applied current).

In this course, we will restrict our attention exclusively to systems of the form (1). We might wonder if this is justified. After all, many ODE models of natural phenomena are not first-order or autonomous. In fact, however, almost any ordinary differential equation can be rewritten as a first-order system of the form (1) after a change of variable. The trade off is that we may have to consider more variables than we started with.

Reduction algorithm:

Step 1: Rewrite the differential equation in the form

$$\begin{aligned} x^{(n)}(t) &= f(t, x(t), x'(t), x''(t), \ldots, x^{(n-1)}(t)) \\ x(0) = x_0,\ x'(0) &= x_0',\ \ldots,\ x^{(n)}(t) = x_0^{(n)}. \end{aligned} \tag{2}$$

Step 2: Assign an independent variable to $x(t)$ and then to each derivative of $x(t)$ up to the $(n-1)^{st}$ order. That is to say, we set

$$x_1(t) = x(t), \quad x_2(t) = x'(t), \quad x_3(t) = x''(t), \quad \text{etc.}$$

Step 3: Recognize that we now have a sequence of first-order differential relationships between the new variables $x_i(t)$, $i = 1, \ldots, n$. In particular, we have that

$$x_1'(t) = x_2(t), \quad x_2'(t) = x_3(t), \quad x_3'(t) = x_4(t), \quad \text{etc.}$$

This gives $n-1$ linear, autonomous differential equations for the variables $x_1(t)$ through $x_{n-1}(t)$.

Step 4: For the remaining equation (for $x_n(t)$), substitute the variable substitutions found in Step 2 into (2) to get

$$x_n'(t) = f(t, x_1, \ldots, x_{n-1}).$$

Step 5: If applicable, substitute the initial conditions $x_1(0) = x(0) = x_0$, $x_2(0) = x'(0) = x_0'$, etc.

Step 6: If the system is non-autonomous, introduce the variable $\tau = t$. This gives the first-order derivative (with initial conditions):

$$\frac{d\tau}{dt} = 1, \quad \tau(0) = 0.$$

The end result of this process, if successfully carried out, is a first-order, autonomous system of differential equations. If the system was **autonomous** to begin with, this system should contain the n state variables $x_1(t)$ through $x_n(t)$ and the form of the system is:

$$\begin{aligned} \frac{dx_i}{dt} &= x_{i+1}, & x_i(0) &= x_0^{(i)} \quad \text{for } i = 1, \ldots, n-1 \\ \frac{dx_n}{dt} &= f(x_1, x_2, \ldots, x_n), & x_n(0) &= x_0^{(n)}. \end{aligned} \tag{3}$$

If the system was **non-autonomous**, there will be the $n+1$ state variables $x_1(t)$ through $x_n(t)$ and τ and the form of the system is:

$$\begin{aligned} \frac{dx_i}{dt} &= x_{i+1}, & x_i(0) &= x_0^{(i)} \quad \text{for } i = 1, \ldots, n-1 \\ \frac{dx_n}{dt} &= f(\tau, x_1, x_2, \ldots, x_n), & x_n(0) &= x_0^{(n)}. \\ \frac{d\tau}{dt} &= 1, & \tau(0) &= 0. \end{aligned} \tag{4}$$

This process can also be generalized to the case where there are more than one original variables of higher-orders to begin with (homework!).

We will see that there are many benefits to considering first-order systems of ordinary differential equations. The main advantage is that first-order derivatives are easy to interpret **geometrically**. A positive derivative means the corresponding variable is increasing, while a negative one means

it is decreasing. Such interpretations are not so convenient for second and third order derivatives.

Having first-order systems will allow us to draw pictures known as vector or direction fields which will tell us a great deal about qualitative behavior without having to search for explicit solutions, which may be impossible. We will investigate this in more detail when we discuss second-order systems in a few weeks. We will find that a picture really is worth a thousand words.

Example 1: Transform the following non-autonomous second-order differential equation into an autonomous system of two first-order differential equations:

$$\theta''(t) + \frac{g}{L}\sin(\theta(t)) = 0, \qquad \theta(0) = 1,\ \frac{d\theta}{dt}(0) = 0.$$

Solution: We make the substitutions $x_1(t) = \theta(t)$ and $x_2(t) = \theta'(t)$. It follows immediately that $x_1'(t) = \theta'(t) = x_2(t)$ (we *always* get this equation for free). The remaining equation comes from the differential equation itself. Rewriting the equation as

$$\theta''(t) = -\frac{g}{L}\sin(\theta(t))$$

and noting that $\theta''(t) = \theta'(t) = x_2'(t)$ and $\theta(t) = x_1(t)$ we have that

$$x_2'(t) = -\frac{g}{L}\sin(x_1(t)).$$

After adjusting initial conditions, the required system is

$$\begin{aligned} \frac{dx_1}{dt} &= x_2, & x_1(0) &= 1 \\ \frac{dx_2}{dt} &= -\frac{g}{L}\sin(x_1), & x_2(0) &= 0. \end{aligned}$$

Example 2: Transform the following non-autonomous second-order differential equation into an autonomous system of three first-order differential equations:

$$\theta''(t) + \frac{g}{L}\sin(\theta(t)) = K\cos(t), \qquad \theta(0) = 1,\ \frac{d\theta}{dt}(0) = 0.$$

Solution: This differential equation corresponds to the non-linear pendulum equation with an addition *forcing* term $K\cos(t)$ (where K is some constant). Nevertheless, we proceed as earlier in making the substitutions $x_1(t) = \theta(t)$ and $x_2(t) = \theta'(t)$ and noticing that, after rewriting the original differential equation, this gives

$$\begin{aligned}\frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= -\frac{g}{L}\sin(x_1) + K\cos(t).\end{aligned}$$

We are not done, however, since the second equation in the system still contains an explicit dependence on t. To remove this dependence, we make the substitution $\tau = t$, differentiate, and append to the original system to get

$$\begin{aligned}\frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= -\frac{g}{L}\sin(x_1) + K\cos(\tau) \\ \frac{d\tau}{dt} &= 1.\end{aligned}$$

As expected, we now have a system of three equations which depend only upon the three state variables x_1, x_2, and τ. This many seem like a small point at this junction in the course, but is will prove to have astonishing powers when we begin to analyse these complicated systems by geometric and numerical means.