MATH 415, WEEK 5:
Two-dimensional Linear Systems

We will start our consideration of higher-dimensional systems by considering the simplest possible case: the first-order autonomous linear system in two variables. We can state such a system in the general form

\[
\begin{align*}
\frac{dx}{dt} &= ax + by, \quad x(0) = x_0 \\
\frac{dy}{dt} &= cx + dy, \quad y(0) = y_0
\end{align*}
\]

(1)

where \(a, b, c, d \in \mathbb{R}\) are fixed constants and \(x_0\) and \(y_0\) are the initial values of \(x\) and \(y\). It is clumsy to write out equations like this every time they are used, so I will favor the matrix expression

\[
\frac{dx}{dt} = Ax, \quad x(0) = x_0
\]

(2)

where

\[
x = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \quad \frac{dx}{dt} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix}, \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \text{and} \quad x_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.
\]

This matrix expression will also allow us to generalize to higher dimensions very easily. Systems of this form should be familiar from Math 319 or Math 320, but we will spend some time reviewing them now since they will form the basis of our consideration of \textit{nonlinear} systems of differential equations, which may have more complicated right-hand sides than (1).

We will once again approach the problem of analyzing the system (1) from two perspectives: an \textit{analytic approach} and a \textit{qualitative approach}. That is, we are interested in both the problem of finding a solution to (1) (i.e. finding functions \(x(t)\) and \(y(t)\) which simultaneously satisfy the given equations) and the problem of drawing meaningful pictures which describe the long-term qualitative dynamics. We will pay particular attention to the correspondence between the two approaches.
1 Two-Dimensional Vector Fields

Before considering how we might derive the analytic solution \((x(t), y(t))\) to such a system, let’s first answer the more basic question of what types of behavior can happen by drawing a picture.

For one-dimensional systems, we have very few choices: solutions could move left, move right, or stay at the same place. For two dimensional systems, the picture is obviously more complicated. For one thing, the picture will be in the two-dimensional \((x, y)\)-plane since there are two state variables rather than one. We can nevertheless make the following observations:

- The system (1) is first-order so that, at every point in the \((x, y)\)-plane we know whether the solution through the point goes right or left \((x'(t) > 0 \text{ or } x'(t) < 0)\) or up or down \((y'(t) > 0 \text{ or } y'(t) < 0)\).

- The equation \(x'(t) = 0\) corresponds to \(ax + by = 0\) or \(y = -(a/b)x\) while \(y'(t) = 0\) corresponds to \(y = -(c/d)x\). In other words, we know exactly where the solution \((x(t), y(t))\) is completely flat \((y'(t) = 0)\) or completely vertical \((x'(t) = 0)\).

In other words, at each point in the \((x, y)\) plane solutions may tend toward one of eight possible directions, which are perhaps most easily identified with their compass points: North (↑), Northeast (↗), East (→), Southeast (↘), South (↓), Southwest (↙), West (←), and Northwest (↖).

The task before us now is to construct a vector field diagram which consists of arrows which point in these primary directions. While a computer program may construct these diagrams by taking a sufficient sample of points (often around 100 points for decent resolution), we will need to be more intuitive in our constructions so we do spend an overwhelming amount of time on rote calculation (especially on exams!). Consider the following examples.

**Example 1:** Draw the vector field for the following system of differential equations

\[
\frac{dx}{dt} = -x + 3y \\
\frac{dy}{dt} = 3x - y.
\]

**Solution:** We can easily determine that

\[
\frac{dx}{dt} = 0 \implies y = \frac{1}{3}x.
\]
and
\[ \frac{dy}{dt} = 0 \implies y = 3x. \]
The question then becomes what happens in the regions between these two lines. When we are above the line \( y = \frac{1}{3}x \) we have
\[ y > \frac{1}{3}x \implies 3y > x \implies x'(t) = -x + 3y > 0 \]
and when we are above the line \( y = 3x \), we have
\[ y > 3x \implies y'(t) = 3x - y < 0. \]
It follows the dominant flow in this region is to the right \((x'(t) > 0)\) and down \((y'(t) < 0)\). If we consider arrows pointing in the dominant directions in all regions, we arrive at the picture given by Figure 1(a).

**Example 2:** Draw the vector field for the following system of differential equations
\[
\begin{align*}
\frac{dx}{dt} &= -x + 5y \\
\frac{dy}{dt} &= -2x + y.
\end{align*}
\]

**Solution:** We can easily determine that \( x' = 0 \) implies \( y = (1/5)x \), and that \( y' = 0 \) implies \( y = 2x \). When we consider the orthants, we end up with a picture that looks something like Figure 1(b).

Figure 1: A rough sketch of the two example systems. Even without solving the equations, we can get some sense about how the solutions behave!
Without even attempting to solve the system of differential equations, we can tell very important things about the types of behaviors we might encounter. It looks like the solutions of the first system originate somewhere in the top-left or bottom-right, pool together, then travel toward either the top-right or the bottom-left. Solutions of the second system, by contrast, appear to spiral around \((0,0)\), although it is unclear whether they approach \((0,0)\) or drift away.

# 2 Analytic solutions

We now consider the question of how to solve a system of the form (1). We start by recalling a few definitions from linear algebra.

**Definition 2.1.** Consider a square matrix \(A \in \mathbb{R}^{n \times n}\). We say that \(\lambda \in \mathbb{C}\) is an **eigenvalue** of \(A\) with associated **eigenvector** \(v \in \mathbb{C}^n\) if

\[ Av = \lambda v. \]  

(3)

Applications of eigenvalues and eigenvectors are far-reaching and varied, and for the most part beyond the scope of this course. The important thing now is to make sure that we remember how to find the eigenvalue/eigenvector pairs for a given matrix \(A\). This involves a little linear algebra (theory and application) but we will be able to make it systematic after a few iterations.

We start by rewriting the expression (3) as

\[ (A - \lambda I)v = 0 \]  

(4)

where \(0\) is the zero vector (i.e. \(0 = (0,0,\ldots,0)\)). This a meaningless equation by itself, since we do not yet know what \(\lambda\) is, but it does tell us that the eigenvalue equation (3) can only be solved if \((A - \lambda I)\) has a non-trivial **nullspace** (or **kernel**). We do have a simple way of characterizing this property. It can happen if and only if we have

\[ \det(A - \lambda I) = 0. \]  

(5)

This actually tells us all we need to know! To determine the eigenvalues and eigenvectors of \(A\), we need to find the values of \(\lambda\) which satisfy the expression (5) (called the **characteristic polynomial** of \(A\)) and then solve the original expression (4) for \(v\).

More importantly, we know that if we have an eigenvalue/eigenvector pair \(\lambda\) and \(v\), we have the ready-made solution to (2) given by

\[ x(t) = ve^{\lambda t}. \]
It is almost trivial to check that this is a solution! We have

\[ x'(t) = \lambda v e^{\lambda t} = A v e^{\lambda t} = A x(t) \]

where we have used the eigenvector equation \( A v = \lambda v \) in the middle step.

Of course, things are not always as straightforward. In general, we have the following complications:

1. there may be multiple eigenvector/eigenvalue pairs (for \( A \in \mathbb{R}^{n \times n} \) there can be up to \( n \) such pairs);
2. some of the eigenvalues may be repeated (in which case generalized eigenvectors must be found); and
3. some of the eigenvalues might be complex (in which case we must use some identities to get a real-valued solution).

Nevertheless, for the two-dimensional system (1) we have only a few non-degenerate possibilities to consider which are summarized by the following result. (In everything below, \( C_1 \) and \( C_2 \) are arbitrary constants to be determined by the initial conditions.)

**Theorem 2.1.** The general solution \( x(t) = (x(t), y(t)) \) to the two-dimensional first-order autonomous system of differential equations given by (1) has the following form:

1. **Two real distinct eigenvalues (or a repeated eigenvalue with two distinct eigenvectors)** - If we have two distinct eigenvalues \( \lambda_1 \) and \( \lambda_2 \) corresponding to \( v_1 \) and \( v_2 \), respectively, the general solution is given by

\[ x(t) = C_1 v_1 e^{\lambda_1 t} + C_2 v_2 e^{\lambda_2 t}. \]

Similarly, if there is a repeated eigenvalue (\( \lambda = \lambda_1 = \lambda_2 \)) but two linearly independent eigenvectors \( v_1 \) and \( v_2 \), we have

\[ x(t) = e^{\lambda t} (C_1 v_1 + C_2 v_2). \]

2. **Repeated eigenvalue, one eigenvector** - If we have a repeated eigenvalue \( \lambda = \lambda_1 = \lambda_2 \) but only one eigenvector \( v \), we have the general solution

\[ x(t) = (C_1 v + C_2 (tv + w)) e^{\lambda t} \]

where \( w \in \mathbb{R}^2 \) is a generalized eigenvector satisfying

\[ (A - \lambda I) w = v. \]
3. **Complex eigenvalues** - If we have a complex eigenvalue \( \lambda = \alpha + i\beta \) corresponding to a complex eigenvector \( \mathbf{v} = a + ib \) then the general solution is given by

\[
\mathbf{x}(t) = C_1 e^{\alpha t} (a \cos(\beta t) - b \sin(\beta t)) + C_2 e^{\alpha t} (a \sin(\beta t) + b \cos(\beta t)).
\]

There are a few notes worth making about these solutions:

1. In all cases, the origin \((x, y) = (0, 0)\) is a fixed point. This is the point relative to which we will measure the stability of the system. Either things will be flowing toward \((0, 0)\) or away from \((0, 0)\).

2. In terms of limiting behavior, exponentials dominate the behavior (i.e. they asymptotically overwhelm the factor \(t\) in case (2), and the trigonometric functions in (3)). Trajectories tend to decay (i.e. approach \((0, 0)\)) if \(\text{Re}(\lambda) < 0\) and blow up (i.e. go away from \((0, 0)\)) if \(\text{Re}(\lambda) > 0\). We will consider the implications of this reasoning more when we consider the qualitative aspects of these solutions.

3. We encounter a degenerate case when \(\lambda = 0\). Although the formula for case (1) works, it is worth noting that this gives rise to a line of equilibrium solutions through \((0, 0)\) so that the qualitative picture is different.

**Example 1:** Determine the solution of

\[
\frac{dx}{dt} = -x + 3y, \quad x(0) = 1
\]

\[
\frac{dy}{dt} = 3x - y, \quad y(0) = 1.
\]

We start by determining the eigenvalues of \(A\). We have

\[
A = \begin{bmatrix} -1 & 3 \\ 3 & -1 \end{bmatrix}, \quad \text{so} \quad (A - \lambda I) = \begin{bmatrix} -1 - \lambda & 3 \\ 3 & -1 - \lambda \end{bmatrix}.
\]

We need to set the determinant of this matrix equal to zero, so that we have

\[
(-\lambda - 1)(-\lambda - 1) - 3(-3) = \lambda^2 + 2\lambda - 8 = (\lambda + 4)(\lambda - 2) = 0.
\]

It follows that there are two eigenvalues, \(\lambda_1 = -4\) and \(\lambda_2 = 2\).
Corresponding to $\lambda_1 = -4$, we have

$$(A - (-4)I) = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$$

We need to find the nullspace of this matrix, so we row reduce

$$\begin{bmatrix} 3 & 3 & 0 \\ 3 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$ We can find the nullspace by setting $v_2 = t$, so that, from the first linear, we have $v_1 = -t$. It follows that

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$ It follows that the eigenvector we want is $v_1 = [-1, 1]^T$.

Corresponding to $\lambda_2 = 2$, we have

$$(A - 2I) = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}.$$ We can row reduce to find

$$\begin{bmatrix} -3 & 3 & 0 \\ 3 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so that $v_2 = t$ implies $v_1 = t$ so that the relevant eigenvector is $v_2 = [1, 1]^T$.

It follows that the general solution is

$$x(t) = C_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-4t} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}.$$ It remains to use the initial conditions to solve for $C_1$ and $C_2$. We have that $x(0) = 1$ and $y(0) = 1$ so that at $t = 0$ we have

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = C_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$ We can rewrite this as

$$-C_1 + C_2 = 1 \quad \text{and} \quad C_1 + C_2 = 1.$$
or, equivalently,
\[
\begin{bmatrix}
-1 & 1 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
C_1 \\
C_2
\end{bmatrix} =
\begin{bmatrix}
1 \\
1
\end{bmatrix}.
\]

We can easily row-reduce this to get
\[
\begin{bmatrix}
-1 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 1
\end{bmatrix}.
\]

It follows that \( C_1 = 0 \) and \( C_2 = 1 \) so that the specific solution is
\[
x(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}
\]
of, equivalently \( x(t) = e^{2t} \) and \( y(t) = e^{2t} \).

It is worth emphasizing now the connection between the solution \( x(t) = (x(t), y(t)) \) and the qualitative vector field diagram (see Figure 2). In particular, recall that vector field diagram does not keep track of time. We can see where a trajectory is going, but we cannot tell how quickly it is moving. To gain this information, it is necessary to plot the individual components of the solution, \( x(t) \) and \( y(t) \). There are advantages and disadvantages to each approach to plotting solutions, but we should familiarize ourself with the connection between the two.

![Figure 2](image_url)

**Figure 2**: The first plot shows solution of (6) overlain on the relevant portion of the vector field the solution traverses from \( t = 0 \) to \( t = 1 \). The second two plots show the components of the solution plotted over the same interval.

**Example 2**: Determine the solution of
\[
\begin{align*}
\frac{dx}{dt} &= x - 4y, & x(0) &= -1 \\
\frac{dy}{dt} &= x - 3y, & y(0) &= 2.
\end{align*}
\]

To find the eigenvalues, we realize

\[ A = \begin{bmatrix} 1 & -4 \\ 1 & -3 \end{bmatrix}, \quad \text{so} \quad (A - \lambda I) = \begin{bmatrix} 1 - \lambda & -4 \\ 1 & -3 - \lambda \end{bmatrix}. \]

The characteristic polynomial is given by

\[(1 - \lambda)(-3 - \lambda) + 4 = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2 = 0\]

so that \( \lambda = -1 \) is a repeated eigenvector. To check for the eigenvector(s) corresponding to this value, we have

\[(A - (-1)I) = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}.\]

To find the nullspace, we row reduce to get

\[
\begin{bmatrix} 2 & -4 & 0 \\ 1 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

so that \( \mathbf{v} = [2, 1]^T \). We notice that we have not obtained two linear independent eigenvectors, so that we need to look for a generalized eigenvector \( \mathbf{w} \). We have

\[(A - (-1)I)\mathbf{w} = \mathbf{v} \implies \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}.\]

If we set \( w_2 = t \), we see that \( w_1 = 1 + 2t \) so that we have

\[
\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 + 2t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \end{bmatrix}.
\]

Setting \( t = 0 \), we have \( \mathbf{w} = [1, 0]^T \).

The general solution is given by

\[
x(t) = \left(C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 \left(t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)\right) e^{-t}
\]

To solve for \( C_1 \) and \( C_2 \), we utilize the initial conditions \( x(0) = -1 \) and \( y(0) = 2 \). At \( t = 0 \) we have

\[
\begin{bmatrix} -1 \\ 2 \end{bmatrix} = C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]
which implies
\[
\begin{bmatrix}
2 & -1 \\
1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
C_1 \\
C_2 \\
\end{bmatrix} = \begin{bmatrix}
-1 \\
2 \\
\end{bmatrix}
\]
so that we have
\[
\begin{bmatrix}
2 & 1 & -1 \\
1 & 0 & 2 \\
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & 2 \\
0 & 1 & -5 \\
\end{bmatrix}
\]
so that \(C_1 = 2\) and \(C_2 = -5\). It follows that the solution is
\[
x(t) = \left(2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} - 5 \left(t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)\right) e^{-t} = \left(\begin{bmatrix} -1 \\ 2 \end{bmatrix} - t \begin{bmatrix} 10 \\ 5 \end{bmatrix}\right) e^{-t}
\]
or, equivalently, \(x(t) = (-1 - 10t) e^{-t}\) and \(y(t) = (2 - 5t)e^{-t}\). For a comparison the vector field and solution plots, see Figure 3.

Figure 3: The first plot shows solution of (7) overlain on the relevant portion of the vector field the solution traverses from \(t = 0\) to \(t = 10\). The second two plots show the components of the solution plotted over the same interval.

**Example 3:** Determine the solution of
\[
\begin{align*}
\frac{dx}{dt} &= -x + 5y, \quad x(0) = 1 \\
\frac{dy}{dt} &= -2x + y, \quad y(0) = 1.
\end{align*}
\]
(8)

To find the eigenvalues, we realize
\[
A = \begin{bmatrix}
-1 & 5 \\
-2 & 1 \\
\end{bmatrix}, \quad \text{so} \quad A - \lambda I = \begin{bmatrix}
-1 - \lambda & 5 \\
-2 & 1 - \lambda \\
\end{bmatrix}.
\]
The characteristic polynomial is given by

\[-1 - \lambda)(1 - \lambda) + 10 = \lambda^2 + 9 = 0.\]

It follows that \(\lambda = \pm 3i\). We need to find the eigenvectors corresponding to these values. We have

\[(A - (3i)I) = \begin{bmatrix} -1 - 3i & 5 \\ -2 & 1 - 3i \end{bmatrix}.\]

To find the corresponding eigenvector, we row reduce to get

\[
\begin{bmatrix}
-1 - 3i & 5 & 0 \\
-2 & 1 - 3i & 0
\end{bmatrix} \xrightarrow{(1+3i)R_1} \begin{bmatrix}
(-1 - 3i)(-1 + 3i) & 5(-1 + 3i) & 0 \\
-2 & 1 - 3i & 0
\end{bmatrix}
\]

\[
\rightarrow \begin{bmatrix}
10 & -5 + 15i & 0 \\
-2 & 1 - 3i & 0
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & -\frac{1}{2} + \frac{3}{2}i & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

so that \(v = (1 - 3i, 2)\). We rewrite this as

\[v = \begin{bmatrix} 1 - 3i \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + i \begin{bmatrix} -3 \\ 0 \end{bmatrix}.\]

We set \(\alpha = Re(\lambda) = 0\) and \(\beta = Im(\lambda) = 3\) and \(a = Re(v) = (1, 2)\) and \(b = Im(v) = (-3, 0)\). It follows that the general solution is

\[x(t) = C_1 \left( \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cos(3t) - \begin{bmatrix} -3 \\ 0 \end{bmatrix} \sin(3t) \right) + C_2 \left( \begin{bmatrix} 1 \\ 2 \end{bmatrix} \sin(3t) + \begin{bmatrix} -3 \\ 0 \end{bmatrix} \cos(3t) \right).\]

To solve for \(C_1\) and \(C_2\), we utilize the initial conditions \(x(0) = 1\) and \(y(0) = 1\). At \(t = 0\) we have

\[
\begin{bmatrix} 1 \\ 1 \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 \begin{bmatrix} -3 \\ 0 \end{bmatrix}
\]

so that we have

\[C_1 - 3C_2 = 1\]

\[2C_1 = 1.\]
It follows immediately that $C_1 = 1/2$ and $C_2 = -1/6$ so we have

$$x(t) = \frac{1}{2} \left( \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cos(3t) - \begin{bmatrix} -3 \\ 0 \end{bmatrix} \sin(3t) \right) - \frac{1}{6} \left( \begin{bmatrix} 1 \\ 2 \end{bmatrix} \sin(3t) + \begin{bmatrix} -3 \\ 0 \end{bmatrix} \cos(3t) \right)$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(3t) + \frac{1}{3} \begin{bmatrix} 4 \\ -1 \end{bmatrix} \sin(3t)$$

or, equivalently, $x(t) = \cos(3t) + \frac{4}{3} \sin(3t)$ and $y(t) = \cos(3t) - \frac{1}{3} \sin(3t)$. For a comparison the vector field and solution plots, see Figure 4.

Figure 4: The first plot shows solution of (8) overlain on the relevant portion of the vector field the solution traverses from $t = 0$ to $t = 10$. The second two plots show the components of the solution plotted over the same interval.

3 Possible qualitative behaviors

In keeping with our emphasis on qualitative behavior rather than explicit solutions, we might ask what the set of all possible behaviors for a linear system (1) is. This is more complicated than the one-dimensional case, where the only (non-trivial) behaviors were convergence or divergence from fixed points. For two-dimensional linear systems, we must also allow spiralling behavior, converge in one direction but divergence in another, and a few non-trivial degenerate cases.

Regardless, because we have the explicit solutions at our disposal, we can convince ourselves that there are only a finite number of qualitatively
distinct cases to check. Furthermore, we know that they only depend on the eigenvalues! We can break things apart something like this (for representative pictures, see Figure 5):

1. **Two real distinct eigenvalues** (or repeated eigenvalues with two distinct eigenvectors)

   (a) If both eigenvalues are positive \((\lambda_1 > 0 \text{ and } \lambda_2 > 0)\) we say \((0,0)\) is an *unstable node* or *source*.

   (b) If both eigenvalues are negative \((\lambda_1 < 0 \text{ and } \lambda_2 < 0)\) we say \((0,0)\) is a *stable node* or *sink*.

   (c) If the eigenvalues have opposite sign, we say \((0,0)\) is a *saddle point*.

2. **Repeated eigenvalue, one eigenvector**

   (a) If the repeated eigenvalue is positive \((\lambda > 0)\) we say \((0,0)\) is a *degenerate source*.

   (b) If the repeated eigenvalue is negative \((\lambda < 0)\) we say \((0,0)\) is a *degenerate sink*.

3. **Complex eigenvalues**

   (a) If the real part of the eigenvalue is positive \((\alpha > 0)\) we say \((0,0)\) is an *unstable spiral* or *source spiral*.

   (b) If the real part of the eigenvalue is negative \((\alpha < 0)\) we say \((0,0)\) is a *stable spiral* or *sink spiral*.

   (c) If the real part of the eigenvalue is zero \((\alpha = 0)\) we say \((0,0)\) is a *center*.

4. **Zero eigenvalue**

   (a) If there is a zero eigenvalue, we say that the system is *degenerate* (there is a line of fixed points through \((0,0)\)).

4 **General conditions**

We can further clarify the connection between these qualitative pictures by considering the general form of the eigenvalues. In general, for the system

\[
\frac{dx}{dt} = A\mathbf{x}(t), \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\]
the eigenvalues can be found by considering the determinant of

\[(A - \lambda I) = \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix}\]

is zero. We consider the characteristic polynomial

\[(a - \lambda)(d - \lambda) - bc = \lambda^2 - (a + d)\lambda + (ad - bc) = 0.\]

The quantities \(a + d\) and \(ad - bc\) are familiar from linear algebra. The first quantity is the trace (the sum of the diagonal elements), and the second is the determinant. If we set \(tr(A) = a + d\) and \(det(A) = ad - bc\) we have the following equations for \(\lambda\):

\[\lambda^2 - tr(A)\lambda + det(A) = 0.\]
Using the quadratic formula gives

$$\lambda = \frac{tr(A) \pm \sqrt{(tr(A))^2 - 4det(A)}}{2}.$$ 

We know that the qualitative picture changes depending on the nature of the eigenvalues and we can see now that this changes as a function of the trace and determinant. In particular, we see that we have the following cases (also see Figure 6):

1. \((tr(A))^2 - 4det(A) > 0\): Two unique eigenvalues (sources, sinks, saddles)
2. \((tr(A))^2 - 4det(A) = 0\): One repeated eigenvalue (nodes)
3. \((tr(A))^2 - 4det(A) < 0\): Complex eigenvalues (spirals, centers)

![Figure 6: Qualitative behavior as it depends on the trace and determinant of \(A\).]