MATH 415, WEEK 9:
Lyapunov Functions, LaSalle’s Invariance Principle, Damped Nonlinear Pendulum

1 Introduction

We have dealt extensively with conserved quantities, that is, systems

\[
\frac{dx}{dt} = f_1(x, y) \\
\frac{dy}{dt} = f_2(x, y)
\]

for which there is quantity \( E(x, y) \) such that, along trajectories \( (x(t), y(t)) \) of the system (1), the value of \( E(x, y) \) is conserved, i.e.

\[
\frac{dE}{dt} = \frac{dE}{dx} \frac{dx}{dt} + \frac{dE}{dy} \frac{dy}{dt} = 0.
\]  

In this case we were guaranteed that trajectories of our system (1) were restricted to level curves of the function \( E(x, y) \).

In most real-world systems, it is unrealistic to expect such a relationship. In physical systems there is energy loss due to friction, air resistance, heat loss, capacitance, and many other factors. We also saw that conservative systems could not permit sources or sinks; many systems, of course, do contain sources and sinks. These are just the first two of many reasons why we should not expect conservative systems to be widespread.

To see what we can conclude when energy is not conserved, let’s reconsider the nonlinear pendulum. In addition to a nonlinear restoring force \( F_{\text{restoring}} = -k \sin(\theta) \) we now add a damping term to the system. It is common and intuitively pleasing to assume that the damping force on an object is proportional to its velocity, i.e. \( F_{\text{friction}} = -c\theta' \). This captures the intuition that the faster you go, the more drag you experience. Combining these two forces gives

\[
F_{\text{total}}(\theta, \theta') = -k \sin(\theta) - c\theta'
\]

where \( \theta \) is the angle the pendulum makes with the vertical axis (positive is counter-clockwise) and \( c, k > 0 \) are two constants. Using Newton’s second law, we arrive at the governing dynamics

\[
m\theta'' = -k \sin(\theta) - c\theta' \quad \Rightarrow \quad m\theta'' + c\theta' + k \sin(\theta) = 0
\]
where \( m > 0 \) is the mass. Making the substitutions \( x_1(t) = \theta(t) \) and \( x_2(t) = \theta'(t) \), we have the system

\[
\begin{align*}
\frac{dx_1}{dt} &= x_2 \\
\frac{dx_2}{dt} &= -\frac{k}{m} \sin(x_1) - \frac{c}{m} x_2.
\end{align*}
\] (3)

It can be seen that the system is nonlinear but not state-dependent or Hamiltonian. Our previous conservation method approach will be apply. We might guess on physical principle, however, that the total energy function

\[ E(x_1, x_2) = \frac{m}{2} x_2^2 - k \cos(x_1) \] (4)

derived from the undamped system may still be applicable. After all, we still have well-define notions of kinetic and potential energy, which compose the total energy function (4).

We now want to see what happens to (4) along solutions of (3). We have

\[
\frac{dE}{dt} = \frac{dE}{dx_1} \frac{dx_1}{dt} + \frac{dE}{dx_2} \frac{dx_2}{dt} = \left(k \sin(x_1)\right) x_2 + \left(m x_2\right) \left(-\frac{k}{m} \sin(x_1) - \frac{c}{m} x_2\right) = -c x_2^2 \leq 0
\]

What the result tell us is that the system is almost always losing energy. In fact, the energy is strictly decreasing (i.e. \( E' < 0 \)) everywhere except for where \( x_2 = 0 \). After a moment’s thought, we realize that we should have expected this. The system loses energy due to friction, but whenever \( x_2 = 0 \) we have no velocity and therefore no damping. In other words, so long as the pendulum is moving it is losing energy.

So what does this mean for our trajectories? Let’s reconsider the contour plot of \( E(x_1, x_2) \) (see Figure 1). Along each level curve of \( E(x_1, x_2) \) our state variables are mapped to the same energy level. It is like an elevation map for a terrain. The new possibility that

\[ \frac{dE}{dt} < 0 \]

tells us that, rather than staying at the same energy level, we may only go downhill. This is exactly what potentials told us for one-dimensional systems, but we will have to be a little more careful than we were there. For
Figure 1: Contour plot of $E(x_1, x_2) = m \frac{x_2^2}{2} - k \cos(x_1)$ with $m = k = 1$.

one-dimensional systems, this was enough to uniquely determine the direction of the solution, since it could only move left or right. This is no longer true since, for two dimensional systems, there are many paths one can take to go down a hill (see Figure 2).

Figure 2: For two-dimensional systems, there is more than one way to travel downhill, which may lead to significantly different long-term behavior.

Let’s reconsider the picture given in Figure 1. While we imagine that our solutions will travel downhill until they reach the bottom at $C = -1$ (which corresponds to a fixed point of (3)), there are many points where $C = -1$. If we start at a sufficiently high level set of $E$ (i.e. high velocity), we cannot tell which local minimum we will end up approaching. Since each such minimum corresponds to the same physical state (i.e. pendulum resting at the bottom of its arc), we may be tempted to disregard this technicality. This just tells us how many times we swing over the top before settling down. For other problems, however, there may be a significant difference between two downhill fixed points. It is also possible, but beyond the scope
of our analysis here, to approach the saddle points where $C = 0$.

To see what can really happen, let’s quickly construct the vector field and perform linear stability analysis. We can see that $x_1' = 0$ implies $x_2 = 0$ as before, but that we now have $x_2' = 0$ implies $x_2 = -\frac{k}{c} \sin(x_1)$. The fixed points, however, are still at $(\bar{x}_1, \bar{x}_2) = (n\pi, 0)$, $n \in \mathbb{Z}$, which should not surprise us. The fixed points for even $n$ correspond to the pendulum hanging down at its resting position, and odd $n$ correspond to the pendulum resting upright. The linearization at these points gives

$$DF(x_1, x_2) = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} \cos(x_1) & -\frac{c}{m} \end{bmatrix}.$$  

For even values of $n$, we have that

$$DF(n\pi, 0) = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}$$

which has the eigenvalues $\lambda_{1,2} = -\frac{c \pm \sqrt{c^2 - 4km}}{2m}$. Regardless of the value of $c$, $k$, or $m$, the real part of the eigenvalue is strictly negative so that $(n\pi, 0)$ is a sink for even $n$. For odd values of $n$, we have that

$$DF(n\pi, 0) = \begin{bmatrix} 0 & 1 \\ \frac{k}{m} & -\frac{c}{m} \end{bmatrix}$$

which has the eigenvalues/eigenvectors pairs $\lambda_{1,2} = -\frac{c \pm \sqrt{c^2 + 4km}}{2m}$. Since $|c| = \sqrt{c^2} < \sqrt{c^2 + 4km}$ we have that there is always one positive and one negative eigenvalue so that $(n\pi, 0)$ is a saddle for odd $n$. The vector field plots with a few representative solutions is contained in Figure 3. As expected, the solutions cut through the level curves given in Figure 1.

![Figure 3: Numerically simulated phase portrait of the damped pendulum.](image-url)
2 Lyapunov Functions

Consider the system

\[
\frac{dx}{dt} = -x^3 + y \\
\frac{dx_2}{dt} = -x - y^3.
\]  \hspace{1cm} (5)

We want to determine the qualitative behavior of the system. Our first objective is to construct the vector field. We can quickly determine that the nullclines are given by

\[ x' = 0 \implies y = x^3 \]

and

\[ y' = 0 \implies y = -\sqrt[3]{x}. \]

After considering the bound regions, the vector field plot can be seen to that given in Figure 4.

![Figure 4: In (a), the vector field plot of the system (5) is presented with the nullclines \( x' = 0 \) and \( y' = 0 \). Solutions appear to swirl in toward \((0,0)\) but the linearization does not allow us to conclude this. In (b), the vector field is overlain with level sets of the decreasing Lyapunov function \( L(x, y) = x^2 + y^2 \).](image)

We can clearly see that solutions appear to spiral toward \((0,0)\). In order to conclude this, however, we need a stronger result than our intuition. The only test at our disposal so far for concluding a fixed point behaves locally like a spiral sink is linear stability analysis. We quickly check that

\[
Df(x, y) = \begin{bmatrix}
-3x^2 & 1 \\
-1 & -3y^2
\end{bmatrix}
\]
so that

\[ DF(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \]

This has purely imaginary eigenvalues \( \lambda = \pm i \), which indicates that the local system behaves like a center rather than the spiral sink we expected. We recall, however, that we can only correspond the behavior of the linear system to the nonlinear one if the fixed point was hyperbolic, which this one is not. In this case, linearization has failed.

We have encountered this situation with a number of previous examples. In the previous cases, however, we were able to show that the system was conservative and conclude that the centers in the linear picture corresponded to \textit{nonlinear centers} in the original system. Our current situation is different: we want to show that the system has a \textit{nonlinear spiral sink}. We cannot conclude this by any analysis we have considered thus far in the course.

Instead, we will build some intuition. We know now that “energy-like” functions can be used to not only restrict solutions to constant energy levels, but they can also be used to force solutions \textit{downhill}. Taking this intuition to its natural conclusion, if we are able to push all solutions downhill for long enough, we should eventually reach the bottom. This well or valley in the energy profile must correspond to a stable fixed point because all solutions are being forced to it.

For systems which do not have a clear mechanical notion of “energy”, the question of which function to choose is a very complicated one. In this case, let’s take the symmetry and uniformity of the picture in Figure 4 into account and pick the function

\[ L(x,y) = x^2 + y^2. \]

This function keeps track of how far trajectories are from \((0,0)\). Taking the derivative along solutions will tell us whether solutions of (5) are staying the same distance from \((0,0)\), approaching, or moving away. We compute

\[
\frac{dL}{dt} = \frac{dL}{dx} \frac{dx}{dt} + \frac{dL}{dy} \frac{dy}{dt} \\
= (2x)(-x^3 + y) + (2y)(-x - y^3) \\
= -2x^4 - 2y^4 < 0.
\]

This tells us that solutions are always approaching \((0,0)\). Unlike the pendulum example, because \((0,0)\) is the unique minimum of \(L\), we can actually conclude that all solutions converge to \((0,0)\) (see Figure 4(b)). So \((0,0)\) is, as we expected, a nonlinear spiral sink.
Things worked out very nicely in this example, but we should take some time to formalize what has happened. We start with a more rigorous notion of stability than we have seen so far. To do this compactly, it will be helpful to introduce a notion from linear algebra, the vector norm or Euclidean distance
\[ \|x - y\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \]
where \( x = (x_1, x_2) \) and \( y = (y_1, y_2) \).

**Definition 2.1.** Consider a fixed point \( \bar{x} = (\bar{x}, \bar{y}) \) of a two-dimensional autonomous system (1). Let \( x(t) = (x(t), y(t)) \) denote the solution with initial condition \( x_0 = (x_0, y_0) \). We will say that \( \bar{x} \) is
1. **stable** if, for every \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that \( \|x_0 - \bar{x}\| < \delta \) implies \( \|x(t) - \bar{x}\| < \epsilon \) for all \( t \geq 0 \).
2. **asymptotically stable** if it is stable and there exists a \( \gamma > 0 \) such that \( \|x_0 - \bar{x}\| < \gamma \) implies \( \lim_{t \to \infty} \|x(t) - \bar{x}\| = 0 \).
3. **unstable** if it is not stable.

These definitions have a lot of math jargon attached to them, but if you do not feel at home in the \( \epsilon - \delta \) definition of things (continuity, limits, etc.), we can re-state the conditions in plain English. A fixed point \( \bar{x} = (\bar{x}, \bar{y}) \) is
1. stable if, when it starts “close” to a fixed point, it stays “close”;
2. asymptotically stable if, when it starts “close” to a fixed point, it converges to the point; and
3. unstable if it is not stable (i.e. if, no matter how “close” we go, there is always some solution which eventually travels “far” away).

It is worth noting that saddle points are unstable by this definition, since there are always solutions close to the point which travel away. That is, we need more than a single trajectory to converge to the point in order for it to be considered stable. With a little thought, we can convince ourselves that nonlinear centers are stable, but not asymptotically stable.

We are interested in classifying fixed points using “conservative-like” methods for systems which are not conservative. The function \( L(x, y) \) used in the previous example provides are archetypal for the class of functions in which we will be interested. They are called Lyapunov functions in honor of Russian mathematician Aleksandr Lyapunov who published work on stability of ordinary differential equations in 1892.
Definition 2.2. Consider a region $U \subseteq \mathbb{R}^2$ which contains a fixed point $(\bar{x}, \bar{y})$ in its strict interior. A function $L(x, y)$ will be called a Lyapunov function in the region $U$ if

1. $L(\bar{x}, \bar{y}) = 0$;
2. $L(x, y) > 0$ if $(x, y) \in U$ and $(x, y) \neq (\bar{x}, \bar{y})$; and
3. $L(x, y)$ is convex (i.e. concave up) in $U$.

This is again more technical than is strictly required. We may restate these conditions in plain English as follows:

1. $L(x, y)$ takes a local minimum at $(\bar{x}, \bar{y})$; and
2. $L(x, y)$ is shaped like a bowl nearby $(\bar{x}, \bar{y})$.

We can see intuitively that these conditions are satisfied for $L(x, y) = x^2 + y^2$. We should, however, provide a check that the function is concave up around $(0, 0)$, since this may not be quite so obvious for general functions. We need to check that $L(x, y)$ has a local minimum at $(0, 0)$, so that we check the second derivative test. We have

$$\left[ \left( \frac{\partial^2 L}{\partial x^2} \right) \left( \frac{\partial^2 L}{\partial y^2} \right) - \left( \frac{\partial^2 L}{\partial x \partial y} \right)^2 \right]_{x=0, y=0} = \left[ (2)(2) - (0)^2 \right]_{x=0, y=0} = 4 > 0$$

and

$$\left[ \frac{\partial^2 L}{\partial x^2} \right]_{x=0, y=0} = 2 > 0.$$

It follows that $L(x, y)$ is concave up around $(0, 0)$. In fact, because the above arguments do not depend on the choice of $(x, y)$, it follows that $L(x, y)$ is a Lyapunov function in the whole $(x, y)$-plane.

The following result is one of the foundational results of stability and control theory.

**Theorem 2.1 (Lyapunov Stability Theorem).** Consider a fixed point $\bar{x} = (\bar{x}, \bar{y})$ and an associated Lyapunov function $L(x, y)$ in a region $U$ around $\bar{x}$. Then $\bar{x}$ is

1. Lyapunov stable if $\frac{dL}{dt} \leq 0$ for all $(x, y) \in U$;
2. asymptotically stable if $\frac{dL}{dt} < 0$ for all $(x, y) \in U$ such that $(x, y) \neq (\bar{x}, \bar{y})$; and
3. unstable if \( \frac{dL}{dt} > 0 \) for all \((x, y) \in U \) such that \((x, y) \neq (\bar{x}, \bar{y})\).

We can clearly see that, for the system (5), \( L(x, y) = x^2 + y^2 \) satisfies condition 2 of Theorem 2.1 so that we may conclude, as expected, that \((0, 0)\) is asymptotically stable. We make the following general notes about Lyapunov functions:

1. **There is no general way to construct Lyapunov functions.** This is a significant difference with conservative mechanical and Hamiltonian systems. These systems gave us methods by which to construct the conserved quantities. No such method exists for systems which have Lyapunov stable fixed points, except in very special cases. It will often be a matter of intuition or trial and error (or a certain all-knowing and benevolent entity giving you hints).

2. When they can be found, Lyapunov functions have the advantage of often explicitly determining the *basin of attraction* for a stable fixed point. We may extend the region \( U \) out for at least as far as \( L(x, y) \) remains concave up—doing so ensures that \( L(x, y) \) is still bowl-shaped and has a unique minimum at \((\bar{x}, \bar{y})\). Solutions have no choice but to slide down the edges of the bowl all the way to the bottom.

3. When they can be found, Lyapunov functions are generally much better suited for handling the stability of systems with undetermined parameter values than linear stability analysis. If you don’t believe me, try computing the eigenvalues and eigenvectors for a matrix with unspecified parameter values. It is a mess.

4. Lyapunov theory historically precedes the Hartman-Grobman Theorem (1892 compared to 1960). Many of the details of the Hartman-Grobman Theorem, in fact, depend upon Lyapunov’s foundation. That the approach remains applicable to modern mathematics and engineering problems over one hundred years after it was introduced should indicate just how powerful (and mysterious) it truly is.

**Example:** Consider the following example drawn from industrial chemistry. The are two species we are interested, a basic *substrate* \( X \) which may combine through a facilitated industrial process to form a new substrate \( Y \) (called a *dimer*). Suppose the basic interactions are as follows:

1. There is constant inflow and outflow into the tank of a substrate \( X \) (a reversible reaction of the form \( \emptyset \rightleftharpoons X \)); and
2. The substrate $X$ combines and disassociates in the tank to form $Y$ in a dimerization reaction (a reversible reaction of the form $2X \rightleftharpoons Y$).

With some normalization of the rate constants, this gives the model

$$\frac{dx}{dt} = 1 - x - 2x^2 + 2y$$
$$\frac{dy}{dt} = x^2 - y. \quad (6)$$

Show that $L(x, y) = x(\ln(x) - 1) + y(\ln(y) - 1) + 2$ is a Lyapunov function for the fixed point $(1, 1)$ and then use this show $(1, 1)$ is asymptotically stable.

[Note: It is necessary to use the inequality $(\alpha - \beta)(\ln(\beta) - \ln(\alpha)) < 0$ for all $\alpha, \beta \in \mathbb{R}$ such that $\alpha \neq \beta$.]

**Solution:** We should first check that this $L(x, y)$ is indeed a Lyapunov function. We check conditions 1-3 of Definition 2.2. We have that

$$L(1, 1) = (1)(\ln(1) - 1) + (1)(\ln(1) - 1) + 2 = (-1) + (-1) + 2 = 0.$$  

It passes the first test. We still need to check that this function has a minimum at $(1, 1)$. We have that

$$\frac{\partial L}{\partial x} = \ln(x) = 0 \implies x = 1$$
$$\frac{\partial L}{\partial y} = \ln(y) = 0 \implies y = 1$$

so that the function has a critical point $(1, 1)$. To check that it is a minimum, we appeal to the second-derivative test, which gives us

$$\left[ \frac{\partial^2 L}{\partial x^2} \right]_{x=1,y=1} = \frac{1}{x} \cdot \frac{1}{y} - (0) = 1 > 0$$

and

$$\frac{\partial^2 L}{\partial x^2} \bigg|_{x=1,y=1} = 1 > 0.$$  

It follows that $(1, 1)$ is a local minimum of $L(x, y)$ so that it is a Lyapunov function in a neighborhood around $(1, 1)$. 


Now we need to consider how the value of $L$ changes along trajectories $x(t) = (x(t), y(t))$ of (6). We have
\[
\frac{dL}{dt} = \frac{dL}{dx} \frac{dx}{dt} + \frac{dL}{dy} \frac{dy}{dt} = \ln(x)(1 - x - 2x^2 + 2y) + \ln(y)(x^2 - y)
\]
\[
= \ln(x)(1 - x) + 2 \ln(x)(y - x^2) + \ln(y)(x^2 - y)
\]
\[
= (\ln(x) - \ln(1))(1 - x) + (\ln(y) - \ln(x^2))(x^2 - y).
\]
By the inequality presented in the hint we can say that the first term is strictly negative unless $1 = x_1$ and that the second term is strictly negative unless $y = x^2$. We therefore have that $L' < 0$ unless $x = y = 1$, i.e. unless we are at the fixed point $(1, 1)$! It follows by Theorem 2.1 that $(1, 1)$ is asymptotically stable (see Figure 5).

Figure 5: Vector field plot of (6) with the level sets of $L(x, y)$ overlain. The value of $L$ decreases along solutions of the system so that they converge to minimum of $L$ at the fixed point $(1, 1)$.

This example should seem extremely unusual and unsettling. After all, how did we know to use this particular function $L(x, y)$, and how did we know to factor in this particular way? Worse still, how did we know to appeal to this particular obscure inequality to complete the argument?

The answer is that many people have been studying (and agonizing) over these examples for many decades, in a great many separate disciplines. It turns out that the function $L(x, y)$ given is a commonly used “energy-like” function in chemical kinetics related to the Gibbs free-energy. The
understanding of this connection, of course, is well beyond the scope of this course. We should, however, feel satisfied that (with a little guidance!) we were able to apply a mathematical tool to a completely new field of science.

3 LaSalle’s Invariance Principle

Let’s briefly reconsider the nonlinear damped pendulum model (3) from earlier. We were able to convince ourselves that, along solutions \( x(t) = (x_1(t), x_2(t)) \), the energy function

\[
E(x_1, x_2) = \frac{m}{2} x_2^2 - k \cos(x_1)
\]

took the form

\[
\frac{dE}{dt} = -cx_2^2 \leq 0.
\]

We would like to formally apply Definition 2.2 and Theorem 2.1 to conclude that the fixed points \((n\pi, 0)\) are asymptotically stable. We cannot conclude this, however, because \( E'(t) \) is not strictly negative nearby \((0, 0)\). The points near \((0, 0)\) where \( E'(t) = 0 \) are enough to invalidate the argument.

We have already noted that the points where \( E'(t) = 0 \) coincide with the points where \( x_2 = 0 \) (i.e. the velocity is zero). The system does not lose energy at these points, but these points clearly do not mean the system stops losing energy altogether; rather, the pendulum will pass through these points and then continue to lose energy. This will continue until the solution is at the bottom of the well.

We would like to include behavior like this in our Lyapunov function framework. We have the following result.

Theorem 3.1 (LaSalle’s Invariance Principle). Consider a fixed point \( \bar{x} = (\bar{x}, \bar{y}) \) and an associated Lyapunov function \( L(x, y) \) in a region \( U \) around \( \bar{x} \). Suppose that:

1. \( \frac{dL}{dt} \leq 0 \) for \((x, y) \in U\); and

2. The only whole solution for which \( \frac{dL}{dt} = 0 \) is \( x(t) = \bar{x} \).

Then \( \bar{x} \) is asymptotically stable.

In other words, so long as the set of points where \( E'(t) = 0 \) is not an invariant set (i.e. you are not trapped there), then you must approach the fixed point.
We can quickly see that this is exactly the intuition we built regarding our energy function $E(x_1, x_2)$. We pass through the points where $x_2 = 0$ and then continue to lose energy.

Formally, we should check the details. We need to check conditions 1-3 of Definition 2.2. We will see there are some subtleties along the way. We will check this for the point $(0, 0)$ and recognize that it is the same for any point $(n\pi, 0)$, $n \in \mathbb{Z}$, where $n$ is even. First of all, we have

$$E(0, 0) = \frac{m}{2} (0)^2 - k \cos(0) = -k \neq 0.$$  

At first glance, this seems bad, but it can be quickly adjusted. The value of the level sets did not matter. It is only the direction of the solutions across the level sets which was of dynamical interest to us. The moral of the story is that we can adjust this function so the minimum value at $(0, 0)$ is $0$ rather than $-k$. To do this we simply add $k$ to the original energy function. The Lyapunov function we need to use is

$$L(x_1, x_2) = \frac{m}{2} x_2^2 - k \cos(x_1) + k \tag{7}$$

so that $L(0, 0) = 0$.

We now check that $L(x_1, x_2)$ has a local minimum at $(0, 0)$. We have

$$\frac{\partial L}{\partial x_1} = k \sin(x_1) = 0 \implies x_1 = n\pi, n \in \mathbb{Z}$$

$$\frac{\partial L}{\partial x_2} = mx_2 = 0 \implies x_2 = 0.$$  

It follows that $(0, 0)$ is a critical point. Applying the second derivative test, we have

$$\left[ \left( \frac{\partial^2 L}{\partial x^2} \right) \left( \frac{\partial^2 L}{\partial y^2} \right) - \left( \frac{\partial^2 L}{\partial x \partial y} \right)^2 \right]_{x=0, y=0} = [(k \cos(x_1)) \left( m \right)]_{x=0, y=0} = km > 0$$

and

$$\left[ \frac{\partial^2 L}{\partial x^2} \right]_{x=0, y=0} = k > 0$$

so that $(0, 0)$ is a local minimum. It follows that there is a region $U$ around $(0, 0)$ where $L(x, y)$ is concave up. It should be noted, however, that unlike the Lyapunov functions for previous systems, this $L(x, y)$ has a local region $U$. The convexity changes throughout the $(x, y)$-plane but is guaranteed to be concave up “near” $(0, 0)$.
We now apply LaSalle’s Invariance Principle (Theorem 3.1). We can easily compute that
\[
\frac{dL}{dt} = cx_2^2 \leq 0
\]
so that condition 1 is satisfied. To satisfy condition 2 we consider the set
\[
V = \{(x_1, x_2) \mid x_1 = \text{arbitrary}, x_2 = 0\}
\]
and need to show that there are no complete solutions lying in this set aside from the fixed point \((0, 0)\) (or, generally, \((n\pi, 0)\)). In order for this to be true, we need \(x_2' = 0\); however, along solutions of the system, we have
\[
\frac{dx_2}{dt} = -\frac{k}{m} \sin(x_1) - \frac{c}{m} x_2.
\]
For any \((x_1, x_2) \in V\) except for the points \((n\pi, 0), n \in \mathbb{Z}\), we have that \(x_2' \neq 0\). Therefore, consequently cannot be whole solutions contained in this set so that condition 2 of Theorem 3.1 is satisfied. We conclude that \((0, 0)\) is asymptotically stable (see Figure 6).

![Figure 6: Vector field plot of (3) near (0, 0) with the set V overlain. Since solutions pass through V, LaSalle’s Invariance Principle guarantees they converge to the local minimum of the energy function E at the fixed point (0, 0), as we expected.](image-url)