MATH 415, WEEK 10:
Limit Cycles, Van der Pol Oscillator, Poincaré-Bendixson Theorem

1 Limit Cycles

Recall that, for one-dimensional systems, the global phase portrait and all qualitatively interesting behavior could be determined by considering the behavior at fixed points and then extending outward. Over the past month, we have seen many two-dimensional systems where this same intuition worked very well. The work required to verify that fixed points were stable, unstable, or centers was often challenging, but once these questions were resolved we were able to construct the phase portrait and explain the qualitative behavior in different regions of the (x, y)-plane.

Now let's consider the system

$$\begin{aligned} \frac{dx}{dt} &= x - y + (-x - y)(x^2 + y^2) \\ \frac{dy}{dt} &= x + y + (x - y)(x^2 + y^2) \end{aligned} \tag{1}$$

from Assignment #4. This is obviously highly nonlinear and therefore difficult to analyze directly. We will side-step that problem for a moment. It can be shown that $(\bar{x}, \bar{y}) = (0, 0)$ is the only fixed point and we can easily determine that the linearized system about $(0, 0)$ is governed by

$$D\mathbf{f}(0, 0) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

which has the complex eigenvalues $\lambda = 1 \pm i$. Since $(0, 0)$ is hyperbolic, it follows by the Hartman-Grobman Theorem that the nonlinear system looks like a *spiral source* near $(0, 0)$.

This is insightful but the picture only extends a short distance from $(0, 0)$; far from from $(0, 0)$ the picture may be different. Without being very rigorous, we can use the intuition that being "far" from $(0, 0)$ just means that we are taking $x^2 + y^2$ to be large. We have the following effective behaviors when $x^2 + y^2$ is large:

$$\frac{dx}{dt} \quad \text{behaves like} \quad -x - y$$

and

$$\frac{dy}{dt} \quad \text{behaves like} \quad x - y.$$

This corresponds to a linear system with eigenvalues $\lambda = -1 \pm i$. In other-words, far from $(0,0)$ the system behaves like a *spiral sink*.

This suggests something very peculiar is happening in the system. We have that the system is repelling locally at $(0,0)$ but that the trajectories may not become unbounded in any direction. In other words, they must stay within a bounded region, but this bounded region necessarily excludes $(0,0)$, which was the only fixed point of the system. This raises the question: *where exactly are the solutions going*?

The answer is something we have already seen before, but in a slightly different form. Consider the function $L(x,y) = x^2 + y^2$. This just keeps track of how far a point is away from $(0,0)$. We want to track how this distance changes along solutions $\mathbf{x}(t) = (x(t), y(t))$ of (1). We have

$$\begin{aligned}
\frac{dL}{dt} &= \frac{dL}{dx}\frac{dx}{dt} + \frac{dL}{dy}\frac{dy}{dt} \\
&= 2x(x - y - (x+y)(x^2+y^2)) + 2y(x + y - (-x+y)(x^2+y^2)) \\
&= 2\left[x^2 - xy - x^4 - x^2y^2 - x^3y - xy^3 + xy + y^2 + x^3y + xy^3 - x^2y^2 - y^4\right] \\
&= 2\left[x^2 + y^2 - x^4 - 2x^2y^2 - y^4\right] \\
&= 2\left[x^2 + y^2 - (x^2+y^2)^2\right] \\
&= 2(x^2+y^2)(1 - (x^2+y^2)).
\end{aligned}$$

That was a bit of work, but we have finally arrived at a point where we can answer our earlier question. We have

$$\begin{cases}
\dfrac{dL}{dt} > 0 \quad \text{if} \quad 0 < x^2 + y^2 < 1 \\[2ex]
\dfrac{dL}{dt} < 0 \quad \text{if} \quad x^2 + y^2 > 1 \\[2ex]
\dfrac{dL}{dt} = 0 \quad \text{if} \quad x^2 + y^2 = 1
\end{cases}$$

In other words, the distance from $(0,0)$ is growing near $(0,0)$, shrinking far away, and staying exactly the same if the distance is exactly equal to one. So solutions are converging to the unit circle, but what exactly are they doing once they get there? There are no fixed points, so trajectories on the circle are always moving. There are only two ways to go, and a glance at the vector field reveals the direction of motion must be counterclockwise.

The behavior is made even more clear by converting the system to polar coordinates (see Assignment #4 solutions). We have that

$$\frac{dr}{dt} = r(1-r)(1+r)$$
$$\frac{d\theta}{dt} = 1 + r^2.$$

As we just reasoned, solutions far from $(0,0)$ approach the center ($r > 1$ implies $r' < 0$), solutions near $(0,0)$ move away ($0 < r < 1$ implies $r' > 0$), and solutions on the unit circle remain there ($r = 1$ implies $r' = 0$). Furthermore, solutions are always spinning counterclockwise ($\theta' > 0$) (see Figure 1).

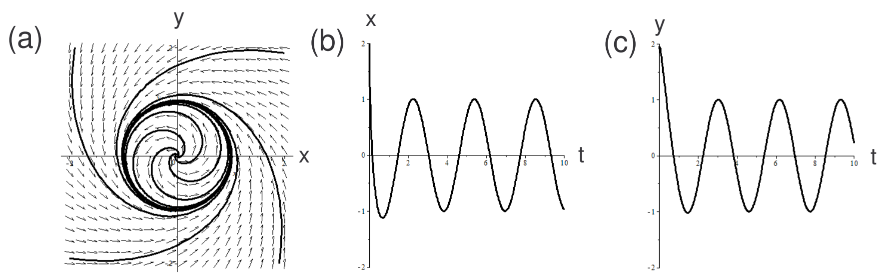

Figure 1: In (a), we have the phase portrait and vector field of system (1). All solutions spiral counterclockwise and converge to the limit cycle with the radius $r = 1$. In (b) and (c), a numerically simulated solution $(x(t), y(t))$ is shown.

We have discovered a completely new qualitative feature of two-dimensional systems: an object which attracts solutions which is *not* a fixed point.

Definition 1.1. *An orbit* $\mathbf{x}(t)$ *of a system of differential equations is called a* ***limit cycle*** *if it is periodic and there are no nearby periodic orbits.*

A key feature of limit cycles, which separate them from the periodic orbits we have encountered for linear and nonlinear centers, is that the periodic orbit is *isolated*. There must be a region about the orbit which does not contained periodic orbits. For two-dimensional systems, topological considerations guarantee that trajectories are either attracting, repelling, or semi-stable, much like fixed points in one-dimensional systems. In other words, we can talk about the stability properties of limit cycles just as we can fixed points.

Stable limit cycles are very important for many physical systems since they indicate a natural rhythm which is *robust to perturbations*. It does not matter if we bump the solutions to the system (1); since the limit cycle is attracting, the solution will find its way back to its natural rhythm. This is a desirable and interesting feature of many naturally cyclic systems, such as biochemical regulation systems, atmospheric circulation, and population dynamics.

2 Van der Pol Oscillator

Consider the following mechanical system, which is known as the *Van der Pol Oscillator*:

$$x'' + \mu(x^2 - 1)x' + x = 0 \tag{2}$$

where $\mu > 0$ is a parameter. The system (2) can be rewritten in the form

$$x'' = -x - (x^2 - 1)x'$$

where we can interpret the right-hand side as a forcing term in a system obeying Newton's second law. The term $-x$ acts like a standard restoring force. The term $-(x^2-1)x'$, however, is very strange. It acts like a standard damping/friction force when x^2 is large but actually *adds* energy to the system when x^2 is small. The equation (2) was originally encountered in studies of electrical circuits but has also found some application in the study of the firing of neurons and seismology.

To analyze the system, we use the substitutions $x_1 = x$ and $x_2 = x'$ to rewrite it in the form

$$\begin{aligned} \frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= -x_1 - \mu(x_1^2 - 1)x_2. \end{aligned}$$

The vector field and some representation solutions for $\mu = 1$ are contained in Figure 2. The limit cycle can be clearly seen; however, unlike the previous example, the cycle does not following a sinuisoidal trajectory. It is instead rather jagged. In fact, there is a unique limit cycle for each value of $\mu > 0$ and the limit cycle has increasingly sharp corners as the value of μ grows.

3 Ruling Out Limit Cycles

If limit cycles are so important to so many applications, the obvious question is: *How do we find them*?

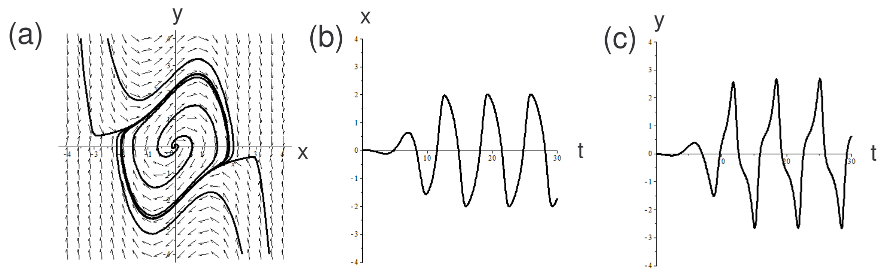

Figure 2: In (a), we have the phase portrait and vector field of system (2). All solutions travel counterclockwise and converge to the irregularly shaped limit cycle in the center. In (b) and (c), a numerically simulated solution $(x(t), y(t))$ is shown.

This is a complicated question which unfortunately has no definitive answer. We do not have simple set of equations to solve as we do with, for instance, fixed points. In fact, it turns out to be much easier to prove that a system *does not* have limit cycles than it is prove that it does. We will start with the following result.

Theorem 3.1. *A system does not have a limit cycle if it satisfies either of the following:*

1. *It is conservative; or*
2. *It admits a global Lyapunov function $L(x, y)$ for which either $L'(t) > 0$ or $L'(t) < 0$ at all but isolated points (i.e. fixed points).*

Intuitive justification of 1.*:* Suppose that a conservative system has an isolated periodic orbit. It follows that, along this orbit the value of the conserved quantity $H(x, y)$ does not change, so that we have $H(\mathbf{x}(t)) = C$ along this solution. Since this periodic solution is isolated, however, because of the topology of the (x, y)-plane, it must be the case that nearby solutions either approach or move away from the orbit asymptotically. This must happen for all nearby solutions so that these solutions also obtain the value $H(\mathbf{x}(t)) = C$ along them. It follows that $H(x, y)$ was not a valid conservation function, and we are done. $\square$

Intuitive justification of 2.*:* To have a limit cycle, it must be the case that there is a $T > 0$ such that $\mathbf{x}(0) = \mathbf{x}(T)$. This implies that $L(\mathbf{x}(0)) = L(\mathbf{x}(T))$

and since the $L'(t)$ is one sign only, it must be the case that $L'(t) = 0$ for all $t \in [0, T]$. Since all such points are isolated in (x, y) by assumption (and we cannot jump from one isolated point to another!), it follows that $\mathbf{x}(t) = \mathbf{x}(0)$ for all $t \in [0, T]$, which means we are at a fixed point. This contradicts the assumption that we had a limit cycle, and we are done. □

These results should not be surprising, but at least they give some teeth to classifications of systems we have studied extensively. For the sake of presenting something new, however, we also introduce the following.

Definition 3.1. *A system will be called a* ***gradient system*** *if there is a function $V(x, y)$ such that*

$$\begin{aligned} \frac{dx}{dt} &= -\frac{\partial V}{\partial x} \\ \frac{dy}{dt} &= -\frac{\partial V}{\partial y}. \end{aligned} \tag{3}$$

Theorem 3.2. *A system is a gradient system if and only if*

$$\frac{\partial f_1}{\partial y} - \frac{\partial f_2}{\partial x} = 0.$$

Proof. We again use equality of mixed order partial derivatives. We have

$$\begin{aligned} & \frac{\partial^2 V}{\partial y \partial x} = \frac{\partial^2 V}{\partial x \partial y} \\ \implies & \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial y}\right) \\ \implies & -\frac{\partial f_1}{\partial y} = -\frac{\partial f_2}{\partial x} \end{aligned}$$

which equals the given expression after rearranging. Since the argument works in both directions, we are done. □

Gradient systems are very similar to Hamiltonian Systems, but we have a different interpretation of them. In particular, the function $V(x, y)$ for gradient systems *do not* represent a conserved quantity. Instead we have the following result.

Theorem 3.3. *Gradient systems do not allow limit cycles.*

Proof. We will suppose that there is a limit cycle, so that there is a $T > 0$ and a solution $\mathbf{x}(t) = (x(t), y(t))$ such that $\mathbf{x}(0) = \mathbf{x}(T)$ and therefore $V(\mathbf{x}(0)) = V(\mathbf{x}(T))$. We now compute the change in V along this solution. We have

$$\begin{aligned} \frac{dV}{dt} &= \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} \\ &= -\left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 \leq 0 \end{aligned}$$

We have to be a little bit careful in our interpretation of this result. Certainly the value of V may never increase along the solution, but it may be zero. In order to be zero, however, we require

$$\frac{dx}{dt} = 0 \quad \text{and} \quad \frac{dy}{dt} = 0.$$

In other words, we may only have $V' = 0$ at *fixed points*. This is incompatible with the assumption that $\mathbf{x}(t)$ is a limit cycle. The result follows. □

Example: Show that the system

$$\begin{aligned} \frac{dx}{dt} &= 2xy + x \\ \frac{dy}{dt} &= x^2 - y^2 \end{aligned} \tag{4}$$

does not permit limit cycles.

Solution: It is sufficient to show that the system is a gradient system. We check

$$\frac{\partial f_1}{\partial y} - \frac{\partial f_2}{\partial x} = \frac{\partial}{\partial y}(2xy + x) - \frac{\partial}{\partial x}(x^2 - y^2) = (2x) - (2x) = 0.$$

It follows from Theorem 3.2 that the system is gradient, and therefore from Theorem 3.3 that it does not permit limit cycles. For completeness, the vector field plot is shown in Figure 3.

4 Poincaré-Bendixson Theorem

We finally turn our attention to *affirming* the existence of limit cycles. This is no trivial task; after all, nonlinear systems are difficult if not impossible to solve in full generality. The strongest tool we have will be the following result, which makes use of the topology of the (x, y)-plane.

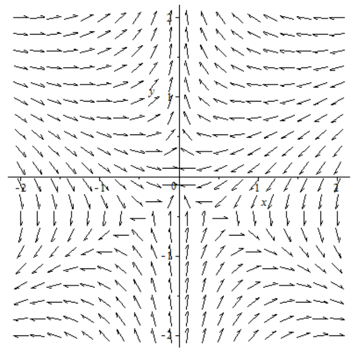

Figure 3: The vector field of the system (4) is shown. The system does not permit limit cycles as a result of it being a gradient system.

Theorem 4.1 (Poincaré-Bendixson Theorem)**.** *Consider a system of autonomous differential equations in two variables. Let R denote a closed, bounded region of the (x, y)-plane which contains no fixed points. Suppose that no solution may leave R. Then the system has a periodic orbit in the region R.*

This result may seem technical but it is exactly the intuition we gave with the system (1) earlier. We reasoned that, if solution could not stay near the fixed point $(0, 0)$ and could not approach infinity, it must be the case that they approached some of limiting stable cycle. This disk-like region, away from $(0, 0)$ but bounded, is the region R in Theorem 4.1. Since trajectories are confined to this region, the result *guarantees* that a limit cycle exists. (It should be noted, however, that this result is uniquely suited to the topology of *two-dimensional systems*. It does not generalize to systems with three or more variables.)

It is common to call the region R a **trapping region** since, if and when any solution enters R it may no longer leave. This is typically accomplished by determine a region where, if we travel along the boundary, we are always pushed *into* the region. The system (1) has a very easy to determine trapping region. We can simply take any disk with a lower radius between 0 and 1, and an upper radius greater than 1, since we are pushed up from the away from $(0, 0)$ at the lower boundary, and toward $(0, 0)$ from the upper boundary. (See Figure 4.)

In general, however, trapping regions may be very difficult to find. Consider the following example.

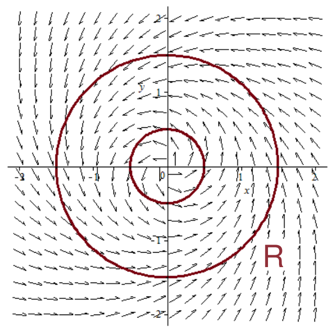

Figure 4: The vector field of the system (1) with the boundaries of an annular trapping region R shown. On the inner circle solutions are forced out, while on the outer circle solutions are forced in.

Example 1: (Chapter 7.3 in text) Consider the following simplified model of *glycolysis*:

1. We track the concentrations of X (ADP) and Y ($F6P$) (i.e. all other involved species are assumed to be plentiful).

2. There is continuous inflow of Y and outflow of X.

3. Y is converted into X in two ways: directly (i.e. a reaction of the form $Y \to X$) and in a facilitated way when there are many molecules of X (in this case, a reaction of the form $2X + Y \to 3X$).

With suitable proportionality constants, this gives rise to the model

$$\begin{aligned} \frac{dx}{dt} &= -x + \frac{y}{10} + x^2 y \\ \frac{dy}{dt} &= \frac{1}{2} - \frac{y}{10} - x^2 y. \end{aligned} \tag{5}$$

Use the Poincaré-Bendixson Theorem to prove that (5) has a limit cycle.

Solution: As is our standard approach by this point, we start by constructing the vector field plot, determining the fixed points, and conducting linear stability analysis. In this case, the nullclines are given by

$$\frac{dx}{dt} = 0 \quad \Longrightarrow \quad y = \frac{10x}{1 + 10x^2}$$

and

$$\frac{dy}{dt} = 0 \quad \Longrightarrow \quad y = \frac{5}{1+10x^2}.$$

These curves are identified in Figure 5(a). We can see that curves intersect one time in the positive orthant. To determine where this point is, we simultaneously solve $x' = 0$ and $y' = 0$ to get

$$\frac{10x}{1+10x^2} = \frac{5}{1+10x^2} \quad \Longrightarrow \quad x = \frac{1}{2} \quad \Longrightarrow \quad y = \frac{10}{7}.$$

We therefore have that the system (5) has a single fixed point at $(1/2, 10/7)$. We perform linear stability analysis at this point to get

$$D\mathbf{f}(x,y) = \begin{bmatrix} -1+2xy & \frac{1}{10}+x^2 \\ -2xy & -\frac{1}{10}-x^2 \end{bmatrix}$$

so that

$$D\mathbf{f}\left(\frac{1}{2}, \frac{10}{7}\right) = \begin{bmatrix} \frac{3}{7} & \frac{7}{20} \\ -\frac{10}{7} & -\frac{7}{20} \end{bmatrix}$$

This has the (rather messy!) eigenvalues

$$\lambda_{1,2} = \frac{11}{280} \pm \frac{\sqrt{27319}}{280}.$$

Since the real part of the eigenvalues is positive, we have that $(1/2, 10/7)$ is a spiral source. It follows that there is a small region around $(1/2, 10/7)$ for which any solution starting outside of the region may not enter.

In order to prove that the system (5) has a limit cycle, we need to show that the solutions which are repelled by the fixed point $(1/2, 10/7)$ stay in a closed and bounded region without fixed points. Since $(1/2, 10/7)$ is the only fixed point, it suffices to show that trajectories stay in a bounded region.

Unlike the previous examples, we do not have a ready-made Lyapunov or distance function with which to bound trajectories. Our approach will see a little unsatisfying for how ad-hoc it may seem to be, but will be very intuitive when considered geometrically. If we consider how the nullclines $x' = 0$ and $y' = 0$ divide the (x, y)-plane in Figure 5, we can see that:

1. trajectories in region 1 (R1) may only flow *up*;
2. trajectories in region 2 (R2) may only flow *right*;

3. trajectories in region 3 (R3) may only flow *down*; and

4. trajectories in region 4 (R4) may only flow *left*.

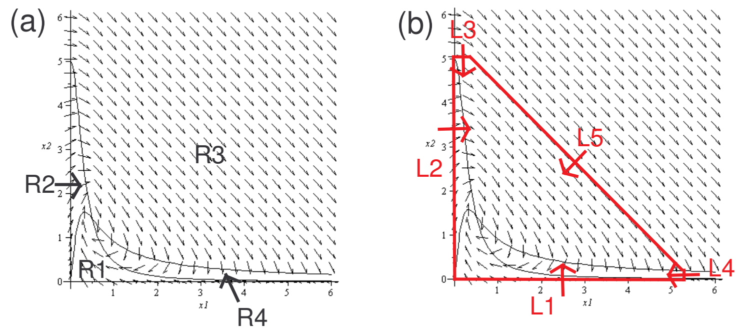

Figure 5: Phase portrait of the system in Example 1. The directional arrows in the four regions in (a) all point in the same direction. The trapping region is given in (b).

This tells us a significant amount about the dynamics. It tells us that solutions may pass through the lines indicated as L1-L4 in Figure 5(b) only from one direction. We will use these lines as boundaries for our trapping region. The only thing remaining to show in this case is that the fifth line ($L5$) does not allow solutions to escape, either. We might observe that once we reach a particular point in region 3 that all the slopes appear to be pointing to the right and down with approximate slope -1. What we hope is that all vectors in fact point *below* the line

$$y = -x + C$$

for some sufficiently enough C. We can actually check this by consider how solutions change relative to the function

$$L(x, y) = x + y$$

at high level sets (since this corresponds to $x + y = C$ for large C). We have

$$\begin{aligned} \frac{dL}{dt} &= \frac{\partial L}{\partial x}\frac{dx}{dt} + \frac{\partial L}{\partial x}\frac{dx}{dt} \\ &= \frac{1}{2} - x. \end{aligned}$$

This is nonnegative so long as we choose $x \geq 1/2$, which means that we only need to extend the line 3 (L3) from $(0, 5)$ out to the point $(1/2, 5)$ and then we can switch to a line of slope negative one and be *guaranteed* that solutions in region 3 do not escape through line 5. Since this closes the region, we have successfully (and rigorously!) constructed a trapping region for trajectories of the system. The Poincaré-Bendixson Theorem guarantees that the system has a limit cycle, which is plotted in Figure 6.

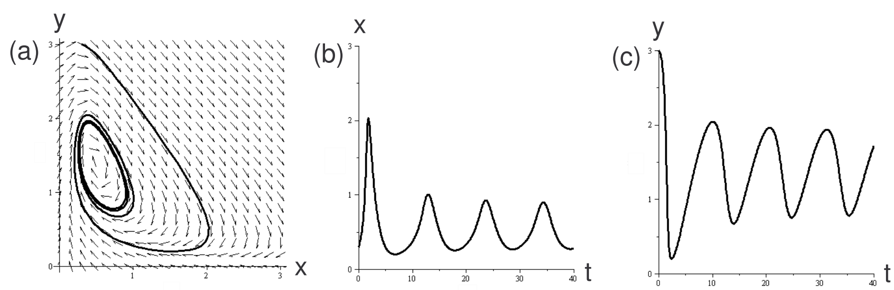

Figure 6: Numerical plots of solutions to the system in Example 1. In (a), we have the solutions in the (x, y)-plane while in (b) and (c) we have the numerically determined functions $x(t)$ and $y(t)$, respectively, plotted as they evolve in time.