MATH 415, WEEK 12 & 13:
Higher-Dimensional Systems, Lorenz Equations, Chaotic Behavior

1 Higher-Dimensional Systems

Consider the following system of differential equations:

\[
\begin{align*}
\frac{dx}{dt} &= 2x - 2y \\
\frac{dy}{dt} &= xy - y \\
\frac{dz}{dt} &= y - z.
\end{align*}
\]

(1)

We can quickly determine that this fits into the general classification of systems we have considered so far in the course. That is, it is an autonomous nonlinear system of ordinary differential equation. We may also be pleased to note that it is only slightly nonlinear, as the only nonlinear term is \(xy\).

The key distinction, of course, is that there are three state variables: \(x, y, \) and \(z\). Our state space is therefore three-dimensional (for the example (1), the state space is the \((x, y, z)\)-plane). We have not attempted to apply any of our nonlinear analysis tools to such systems before, but we will see that the vast majority of existing tools can be easily recalibrated to handle such cases. We quickly go through them now.

1. Vector Field Diagram: We will still determine the nullclines exactly as before. For the system (1), we have

\[
\begin{align*}
x' &= 0 \implies y = x \\
y' &= 0 \implies x = 1 \text{ or } y = 0 \\
z' &= 0 \implies z = y.
\end{align*}
\]

(2)

The intuition is exactly the same as for two-dimensional systems. (e.g. nullcline \(x' = 0\) is where motion is only allowed in the \(y\) and \(z\) variables). However, we should be able to quickly convince ourselves that it is not practical or insightful to actually attempt to plot the three curves given above in the \((x, y, z)\)-plane. This is a general feature of the
analysis of higher-dimensional systems: our geometric “picture” will fail us, although we will sometimes be able to imagine two-dimensional pictures as being embedded in the larger picture (stay tuned!).

2. Fixed Points: We can again determine the fixed points exactly as before. We need to set \( x' = 0, \) \( y' = 0, \) and \( z' = 0. \) If we take \( y = 0 \) in the second equation of (2), we must have \( x = 0 \) from the first and \( z = 0 \) from the last, while if we take \( x = 1 \) in the second equation, we must have \( x = 1 \) from the first and \( z = 1 \) from the last. So the system (1) has two fixed points: \((0, 0, 0)\) and \((1, 1, 1)\).

The interpretation remains the same as previously: these points are invariable to the forces of the system, and may have attracting and repelling stability properties for nearby solutions. It is worth noting, however, that the system of equations (2) were heavily engineered to work out favorably. Solving three nonlinear equations in three variables can often be extremely algebraically challenging.

3. Linear Stability Analysis: We can directly extend the the notion of linearization about fixed points by analyzing a higher-dimensional Jacobian matrix. For the system (1), we have

\[
Df(x, y, z) = \begin{bmatrix}
2 & -2 & 0 \\
y & x - 1 & 0 \\
0 & 1 & -1 \\
\end{bmatrix}
\]

so that

\[
Df(0, 0, 0) = \begin{bmatrix}
2 & -1 & 0 \\
0 & -1 & 0 \\
0 & 1 & -1 \\
\end{bmatrix}.
\]

To determine the eigenvalues, we expand

\[
\det[Df(0, 0, 0) - \lambda I] = \begin{vmatrix}
2 - \lambda & -1 & 0 \\
0 & -1 - \lambda & 0 \\
0 & 1 & -1 - \lambda \\
\end{vmatrix} = 0.
\]

This gives

\[
(2 - \lambda)[(-1 - \lambda)(-1 - \lambda) - (-1)(0)] = (2 - \lambda)(1 + \lambda)^2 = 0.
\]

It follows that \( \lambda_1 = 2 \) and \( \lambda_{2,3} = -1. \) It can also be determined that the eigenvector associated with \( \lambda_1 = 2 \) is \( \mathbf{v}_1 = (0, 0, 1) \) and that \( \lambda_{2,3} = -1 \) has the eigenvector \( \mathbf{v}_2 = (1, 0, 0) \) and generalized eigenvector \( \mathbf{w} = (0, 1, 0). \) Our interpretation is that, near \((0, 0, 0)\) in the \((x, y, z)\)-plane, we have:
Figure 1: Linearized picture of the system (1) near the fixed points (0, 0, 0) and (1, 1, 1).

(a) In the z-direction, solutions are repelled from (0, 0, 0).
(b) In the x and y directions, solutions behave like a degenerate sink node, with dominant mode in the x direction.
(c) In the remaining space, solutions exhibit a combination of these two behaviors.

At the other fixed point, (1, 1, 1), we have

\[
Df(1, 1, 1) = \begin{bmatrix} 2 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}.
\]

To determine the eigenvalues, we expand

\[
\det[Df(1, 1, 1) - \lambda I] = \begin{vmatrix} 2 - \lambda & -2 & 0 \\ 1 & -\lambda & 0 \\ 0 & 1 & -1 - \lambda \end{vmatrix} = 0.
\]

This gives

\[
(2 - \lambda)(-\lambda)(-1 - \lambda) - (-2)(1)(-1 - \lambda) = (-1 - \lambda)(\lambda^2 - 2\lambda + 2) = 0.
\]

It follows that \(\lambda_1 = -1\) and \(\lambda_{2,3} = 1 \pm i\). It can furthermore be determined that the eigenvector associated with \(\lambda_1 = -1\) is \(v_1 = (0, 0, 1)\) and the real and imaginary part of the eigenvectors associated with \(\lambda_{1,2} = 1 \pm i\) are \(a = (1, 2, 1)\) and \(b = (3, 1, 0)\), respectively. Our interpretation is that, near (1, 1, 1) in the \((x, y, z)\)-plane, we have:
(a) In the $z$-direction, solutions approach $(1, 1, 1)$ asymptotically.
(b) In the plane spanned by $(1, 2, 1)$ and $(3, 1, 0)$, the system behaves like a spiral source.
(c) In the remaining space, solutions exhibit a combination of these two behaviors.

A surprising consequence of this analysis is that we do not need any new classifications—we just need to worry how these smaller pictures fit together in more dimensions!

2 Numerical Methods

We have seen that, although the details might be messy, the linear picture near fixed points generalizes in a very nice way from systems with two state variables to those with three. Far from the fixed points, however, the non-linear terms may have a significant influence. Unfortunately, unlike with our two-dimensional systems, we do not generally have a well-defined picture to tell us what is happening in these regions.

This difficulties necessitate an increased reliance on numerical methods, and computers in general, in the analysis of higher-dimensional systems. The good news is that the methods developed for one-dimensional autonomous systems generalize very easily. If we directly approximate the derivatives in the general system of differential equations with three state variables we get

\[ \begin{align*}
x' &= f_1(x, y, z) \implies x(t+\Delta t) &\approx x(t) + f_1(x(t), y(t), z(t))\Delta t \\
y' &= f_2(x, y, z) \implies y(t+\Delta t) &\approx y(t) + f_2(x(t), y(t), z(t))\Delta t \\
z' &= f_3(x, y, z) \implies z(t+\Delta t) &\approx z(t) + f_3(x(t), y(t), z(t))\Delta t.
\end{align*} \]

This suggests the multi-dimensional first-order Euler method

\begin{equation}
\begin{align*}
x_{n+1} &= x_n + f_1(x_n, y_n, z_n)\Delta t \\
y_{n+1} &= y_n + f_2(x_n, y_n, z_n)\Delta t \\
z_{n+1} &= z_n + f_3(x_n, y_n, z_n)\Delta t.
\end{align*}
\end{equation}

That is to say, from the initial condition $(x_0, y_0, z_0)$, we can determine the approximate next point $(x_1, y_1, z_1)$ by substituting into the three equations in (3), and then use that point to find $(x_2, y_2, z_2)$, and so on.

Of course, we know from our study of one-dimensional systems that the first-order Euler method is terrible at bounding the accumulated error of the
process. In practice, we will use the fourth-order Runge-Kutta method. The code is provided on the website and will not be summarized here. A numerical integration of the system (1) is provided in Figure 2.

![Graphs](image)

Figure 2: Numerical solution for (1) evaluated from the initial condition \((x_0, y_0, z_0) = (1.01, 1.01, 1)\) near the fixed point \((1, 1, 1)\). The short-term behavior is that of a tilted spiral source, although the tilt is not evident in these plots. The long-term behavior of the solution is \(x(t) \to -\infty, y(t) \to 0,\) and \(z(t) \to 0\) (not shown).

### 3 Lorenz Equations

At this point, it might be tempting to conclude that three-dimensional systems are merely complicated two-dimensional systems. We have the same tools, and are able to apply them in the same way as two-dimensional systems, to give the same conclusions.

This conclusion would be premature. While we did not dwell on it, the topology of the \((x, y)\)-plane factored significantly in our analysis of the dynamics of two-dimensional systems. In particular, curves in the \((x, y)\)-plane always had the effect of dividing the state space in an orientated way such we could not cross back to the other side. This was the key factor, in particular, in the argumentation for the Poincaré-Bendixson Theorem. Once we trapped solutions, they had no choice but to swirl in one direction or the other until they settled down to a limit cycle. There was simply nowhere else to go: the solutions could not “double back”, so they converge to a limit.

In three or higher dimensions, however, this topological property no longer holds. Solutions do not divide the state so that, at least in principle, we can imagine a solution wandering around in the \((x, y, z)\)-plane and doubling back on itself whenever it wants. There is a lot of space in three-
dimensions—we could imagine it doubling back an infinite number of times, and still having plenty of room to sneak through again. That is to say, solutions may mix in ways which were not previously possible.

But is this really possible, given how well-behaved the systems we have seen are? And if it is possible, what exactly does it mean, anyway? We start by consider the following system of non-linear equations

\[
\begin{align*}
\frac{dx}{dt} &= \sigma (y - x) \\
\frac{dy}{dt} &= rx - y - xz \\
\frac{dz}{dt} &= xy - bz
\end{align*}
\]

where \(\sigma, r, b > 0\) are parameters. This set of equations, know as the Lorenz system, was original derived by Edward Lorenz in 1962 as a model of atmospheric convection. Chapter 9.1 of the text gives a detailed derivation of the equations in a different physical context, that of a rotating water wheel. The Lorenz system is one of the most studied systems in the history of the field of dynamical systems.

We will begin by investigating the local behavior of (4) near fixed points. We can quickly determine that

\[
\begin{align*}
x' &= 0 \quad \Rightarrow \quad y = x \\
y' &= 0 \quad \Rightarrow \quad y(r - 1 - z) = 0 \quad \Rightarrow \quad y = 0 \text{ or } z = r - 1 \\
z' &= 0 \quad \Rightarrow \quad y = \pm \sqrt{b}z \quad \Rightarrow \quad z = 0 \text{ or } y = \pm \sqrt{b(r - 1)}.
\end{align*}
\]

It follows that the fixed points are

\[
\bar{x}_1 = (0, 0, 0), \quad \bar{x}_2 = (-\sqrt{b(r - 1)}, -\sqrt{b(r - 1)}, r - 1), \quad \bar{x}_3 = (\sqrt{b(r - 1)}, \sqrt{b(r - 1)}, r - 1).
\]

It is clear that \(\bar{x}_2\) and \(\bar{x}_3\) only exist if \(r > 1\). It seems reasonable to begin our analysis in the region \(0 < r \leq 1\) since there is only the single fixed point \(\bar{x}_1 = (0, 0, 0)\).

**Case 1** \((0 < r < 1)\): We will approach the stability of \(\bar{x}_1\) through two approaches, linear stability analysis and a Lyapunov function. Firstly, we linearize the system to get

\[
Df(x, y, z) = \begin{bmatrix}
-\sigma & \sigma & 0 \\
r - z & -1 & -x \\
y & x & -b
\end{bmatrix}
\]
so that at $\bar{x}_1$ we have

$$Df(0, 0, 0) = \begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{bmatrix}.$$ 

This has eigenvalues

$$\lambda_1 = -b,$$

$$\lambda_{2,3} = \frac{-\sigma - 1 \pm \sqrt{(\sigma + 1)^2 + 4\sigma(r - 1)}}{2}.$$ 

So long as $0 < r < 1$ we have that $|\sigma - 1| < \sqrt{(\sigma + 1)^2 + 4\sigma(r - 1)}$ so that the real part of every eigenvalue is negative. We may therefore conclude that $\bar{x}_1 = (0, 0, 0)$ is linearly stable.

We might still wonder what happens far from $(0, 0, 0)$. To handle this, we consider the Lyapunov function

$$L(x, y, z) = \frac{x^2}{2\sigma} + \frac{y^2}{2} + \frac{z^2}{2}.$$ 

This function is essentially a three-dimensional bowl with a center at $(0, 0, 0)$. The level curves in three-dimensions are ellipsoids (multi-dimensional ovals). We can quickly compute that

$$\frac{dL}{dt} = \frac{\partial L}{\partial x} \frac{dx}{dt} + \frac{\partial L}{\partial y} \frac{dy}{dt} + \frac{\partial L}{\partial z} \frac{dz}{dt}$$

$$= \frac{x}{\sigma} [\sigma(y - x)] + y (rx - y - xz) + z (xy - bz)$$

$$= -x^2 + (r + 1)xy - y^2 - bz^2.$$ 

The only term which could possibly contribute to the derivative being positive is the cross term $(r + 1)xy$. To show that this is not a concern, we complete the square

$$\frac{dL}{dt} = -\left( x^2 - (r + 1)xy + \left( \frac{r + 1}{2} \right)^2 y^2 - \left( \frac{r + 1}{2} \right)^2 y^2 \right) - y^2 - bz^2$$

$$= -\left( x - \frac{r + 1}{2} y \right)^2 - \left( 1 - \left( \frac{r + 1}{2} \right)^2 \right) y^2 - bz^2.$$ 

We can now see that $L'(t) < 0$ for $(x, y, z) \neq (0, 0, 0)$ whenever $r < 1$ so that $(0, 0, 0)$ is globally asymptotically stable in this region.
This analysis gives us fairly significant foundation from which to move forward. We are starting with the most stable situation imaginable—a single, globally stable fixed point. We now investigate what happens as we allow the parameter values to vary, although we will omit most of the computational details.

**Case 2** \(1 < r < r_H = \frac{\sigma(\sigma + b + 3)}{\sigma - b - 1}\): As we pass through the point \(r = 1\) into the region \(r > 1\) we go from one fixed points to three. We also lose the stability of the fixed point \(\bar{x}_1 = (0, 0, 0)\) because the sign of one of \(\lambda_2\) or \(\lambda_3\) will be positive. The bifurcation value \(r = 1\) is our first hint at exotic behavior for this system: it is a pitchfork bifurcation. It is possible, although difficult, to determine that immediately after the bifurcation the fixed points \(\bar{x}_2\) and \(\bar{x}_3\) are linearly stable.

Our analysis, however, is far from complete. While \(\bar{x}_2\) and \(\bar{x}_3\) are linearly stable near the pitchfork bifurcation at \(r = 1\), the stability still depends upon the rest of the parameters. It can be checked that, for the value \(r_H = \frac{\sigma(\sigma + b + 3)}{\sigma - b - 1}\) the Jacobian matrices \(Df(x_2)\) and \(Df(x_3)\) have a neutrally stable complex eigenvalue. It is significantly easier to verify this result than it is to derive it! With reasonable ease, we can check that the characteristic polynomial corresponding to the linearization at \(\bar{x}_2\) and \(\bar{x}_3\) is

\[
\lambda^3 + (\sigma + b + 1)\lambda^2 + (r + \sigma)b\lambda + 2b\sigma(r - 1) = 0.
\]  \(\text{(5)}\)

A more challenging, but manageable, computation shows that the strictly complex number \(\lambda = i\omega\) for an appropriate \(\omega > 0\) is a solution of (5) and therefore an eigenvalue. (Homework!)

Intuition suggests the following:

1. To one side of the bifurcation value \(r = r_H\), the eigenvalue has negative real part. Based on our previous analysis, we suspect the region to be \(1 < r < r_H\). This suggests stability for \(\bar{x}_2\) and \(\bar{x}_3\) in this region.

2. On the other side of the bifurcation value \(r = r_H\), the eigenvalue has a positive real part. Again, we suspect the region to be \(r > r_H\). This suggests instability for \(\bar{x}_2\) and \(\bar{x}_3\) in this region.

**Case 3** \(r > r_H = \frac{\sigma(\sigma + b + 3)}{\sigma - b - 1}\): When we pass from \(0 < r < 1\) into \(r > 1\) we lose stability of our lone fixed point but gain two stable fixed points. When we pass from \(1 < r < r_H\) into \(r > r_H\), however, we lose the
Figure 3: Simplified bifurcation diagram for the Lorenz system (4). In the region $0 < r < 1$, there is the single globally stable fixed point $\bar{x}_1$. The system undergoes a pitchfork bifurcation at $r = 1$ so that in the region $1 < r < r_H$, there are two stable fixed points $\bar{x}_2$ and $\bar{x}_3$ while $\bar{x}_1$ is unstable. At $r = r_H$, the system undergoes a Hopf bifurcation as the fixed points $\bar{x}_2$ and $\bar{x}_3$ absorb unstable limit cycles and consequently lose stability.

stability of these fixed points. This begs the question of, if the solutions do not tend toward the fixed points, where exactly they do go.

There are several possibilities:

1. *Can solutions become unbounded?* This seems reasonable enough, especially given the instability at fixed points, but it can be shown that the solutions must remain in a bounded region (homework!).

2. *Can there be a stable limit cycle?* This seems reasonable for a number of reasons. Firstly, the fact that solutions do not tend toward fixed points or infinite implies the existence of a trapping region. In two-dimensions, this was sufficient to imply a periodic orbit by the Poincaré-Bendixson Theorem. Furthermore, we have a complex eigenvalue, which suggests spiralling behavior.

Unfortunately, all this intuition turns out to be *wrong*. While the changing of sign in the complex eigenvalue does correspond to a Hopf bifurcation, it can be shown to occur in the wrong direction. That is to say, rather than the bifurcation creating a stable limit cycle in the region $r > r_H$, it absorbs an unstable limit cycle in the region
Figure 4: Numerical integration of the Lorenz system (4) is shown in (a) for parameter values $\sigma = 10$, $b = 8/3$, and $r = 28$ with initial condition randomly chosen near $(0,0,0)$. Projection of the trajectories onto the $(x,z)$ plane is shown in (b). Trajectories swirl around the fixed points on the left and right for a seemingly random amount of time before randomly switching to the other side.

$r < r_H$. (The argument for this property is extremely difficult—even too difficult to be left as homework.)

This analysis leaves us with the somewhat unsatisfying picture given in Figure 3. In the region $r > r_H$, we know that solutions remain bounded, but we have systematically ruled out the possibility that they converge to any of the stable objects we are used to seeing: namely, fixed points and limit cycles. This begs the question of where they actually go. What kind of stable objects are left?

Fortunately, when determining where solutions go in the limit $t \to \infty$, we do have one tool left—we can numerically integrate the system. It can be checked that the parameter values $\sigma = 10$, $b = 8/3$, and $r = 28$ satisfy $r > r_H$. Simulating the system gives the picture in Figure 4. Consistent with our expectations, we can see that solutions remain in a bounded set, do not approach any of the fixed points, and do not settle into a periodic limit cycle. How to classify exactly what solutions are doing, however, is a more perplexing matter!

4 Chaos

Imagine being the first to observe the behavior exhibited by the simulation in (4) and asked to define it. It should not take us long to agree that, as
imperfect as it is, there may be no better description than to call it chaotic behavior. It is almost as if every time the system appears to settle in a regular, predictable rhythm, somebody comes along and shakes the system. Furthermore, the time at when the system is shaken seems to be random. The system defies prediction.

It is one thing to describe things colloquially as chaotic, however, and quite another define them as chaotic mathematically. We have the following.

**Definition 4.1.** A system is said to exhibit chaotic behavior if:

1. solutions are bounded;
2. solutions are aperiodic (except for fixed points); and
3. solutions are extremely sensitive to initial conditions.

We should be pleased to realize that the definition captures exactly the intuition we built when analyzing the Lorenz system. Once we have removed all sources of stability in the system, whatever remains, no matter how strange, must be chaos.

There is a point which is worth stressing between the colloquial definition of chaos and Definition 4.1. In popular culture, chaos is often associated with any system for which subtle changes in initial conditions can lead to wildly different long-term behaviors. A popular example is the so-called “butterfly effect,” whereby it is often supposed that the small atmospheric perturbation of a butterfly flapping its wings in one part of the world might be the difference between a hurricane forming a month later, or not. Such analogies are applied broadly to weather models, which are notoriously inaccurate on time-frames longer than a few days. (Although, in fairness, we should probably spare the poor butterflies from absorbing too much of the blame!)

A result of this difficulty in prediction interpretation of chaos is that it often gets associated with random or stochastic processes—that is, systems, which have random fluctuations. It is tempting to look at a plot like Figure 4 and associate it with white noise or some other random fluctuating process. **This is fundamentally incorrect!** There is a significant difference between the following two statements:

1. Small changes in initial conditions can be lead to wildly different long-term behaviors; and
2. The behavior is random.
Even though both of these statements are consistent with systems which defy prediction—and solutions which look like those in Figure 4—the models we are studying are fundamentally deterministic. As soon as we specify the initial condition, we have exactly one solution which describes the behavior from that point forward. If we run the system a thousand times, each time we will receive the same outcome. As counterintuitive as it, we can think of chaotic behavior is deterministic behavior which looks random.

There is a further note we should make about chaotic systems, which relates to how we may use these models to make predictions in practice.

- **Ideal world** - In a perfect world, we could obtain a solution (either explicitly or numerically) and use it to determine the future state of the system at any time. Although the solutions behave chaotically, if we know the initial state, we may fully predict all future states as a result of the system being deterministic.

- **Real world** - Nonlinear systems can rarely be solved explicitly and we therefore have to rely on some numerical method. This numerical method will have inherently finite precision so that the numerical error will accumulate with each step. If the system is chaotic, this will inevitably lead to a divergence in finite time with the actual solution. That is to say, no matter how good of a numerical method we choose, or how small we take our time step $\Delta t$ to be, our numerical solution will become completely unreliable in finite time. Because of the divergence of nearby solutions, chaos destroys our ability to make accurate predictions about the future.

5 **Strange Attractor**

Chaotic behavior raises many interesting questions which we will not have the time to address in any depth in this course. For the most part, we will content ourselves to *know it when we see it*. That is, we will simulate solutions and check for the characteristic irregularity present in chaotic behavior. We will, however, give some further thought to one particularly thorny theoretical point: *If solutions do not limit to a fixed point of limit cycle, where exactly do they go?*

This should seem like a silly question at first glance. We have spent the previous lectures convincing ourselves that chaotic solutions are always in motion, never settle down, and are essentially unpredictable. But this takes a narrow view of what the limit of a solution is. For those who have taken
Math 521 or another mathematical analysis course, you know that there are many notions of convergence which are of interest, and one of the most useful is subsequential convergence.

Imagine the following situation. We have a solution which behaves chaotically. It dances around within our state space, never repeating itself, and never seeming to obey any set pattern or rhythm. Despite this seeming randomness, suppose we can identify a point in the state space with the following property:

- No matter how far along the solution we go, we will always pass as close to the point as we like at some future time.

Any such point would be, in some weak sense, an attractor of the solution. After all, the solution could never fully break free from such a point. It would always return. Now imagine taking the set of all such points for a chaotic system. What would that set look like?

Before we consider this question, we formally define the following concept, which is foundational to the theoretical study of dynamical systems.

**Definition 5.1.** A point \( x \in \mathbb{R}^n \) is called an \( \omega \)-limit point of a solution \( x(t) \) if there exists a sequence of times \( \{t_n\}_{n=1}^{\infty} \) such that

1. \( \lim_{n \to \infty} t_n = \infty \); and
2. \( \lim_{n \to \infty} x(t_n) = x \).

The set of \( \omega \)-limit points for a particular trajectory \( x(t) \) is called the \( \omega \)-limit set of the solution and is denoted by \( \omega(x(0)) \).

This definition perfectly captures our previous requirement: an omega limit point is a point which is repeatedly approached by a solution \( x(t) \). The difference between this and our standard notion of stability is that we no longer require the solution to actually settle down to the limiting object (a fixed point of limit cycle). It can do as it wishes for as long as it likes, so long as it eventually returns.

We can quickly check that the stable objects we have discussed earlier are \( \omega \)-limit points:

1. **Stable fixed points** - If solutions converge asymptotically to a fixed point \( \bar{x} \), we can choose any sequence of times \( \{t_n\}_{n=1}^{\infty} \) for which \( \lim_{n \to \infty} t_n = \infty \) and are guaranteed that \( \lim_{n \to \infty} x(t_n) = \bar{x} \).
2. **Stable limit cycles** - If solutions asymptotically approach a limit cycle with period $T$, we may take any sequence of points $\{t_n\}_{n=1}^{\infty}$ which successively increment by $T$ (e.g. $\{0, T, 2T, 3T, \ldots\}$). The subsequence of points $x(t_n)$ will approach a point on the limit cycle, and if we take all points for which such a subsequence can be found, we will collect the entire limit cycle.

By definition, the $\omega$-limit set for chaotic solutions may not contain either of these objects. For those of you who have taken Math 521, however, you know that this is not the end of the story. Our solutions are forced to remain in a bounded subset of $\mathbb{R}^n$. One of the classical results from analysis is the **Bolzano-Weierstrass** which tells us that any bounded sequence in $\mathbb{R}^n$ has a convergent subsequence. In other words, every chaotic solution must have a non-empty $\omega$-limit set. We therefore have the following third possibility.

3. **Strange attractor** - If solutions are bounded, but do not approach a fixed point or limit cycle, they approach a non-empty aperiodic, connected, fractal set known as a strange attractor. For each point $x \in \mathbb{R}$ on the strange attractor, the sequence of times $\{t_n\}_{n=1}^{\infty}$ for which $x(t_n)$ converges to $x$ is not regular, periodic, or otherwise predictable.

Unfortunately, it is one thing to guarantee the existence of a strange attractor, and another to determine what it looks like. We have some intuition from earlier that chaos is the result of “mixing” of solutions and guess that this should be reflected in the shape of the strange attractor. Generally, we will resort to numerical methods. A numerical approximation of the strange attractor for the Lorenz system (4) is given in Figure 5.
Figure 5: Numerical approximation of the “strange attractor” for the Lorenz system (4).