Problems for submission:

1. Prove that $P \implies Q$ is equivalent to $\neg Q \implies \neg P$. That is to say, prove that

   $$(P \implies Q) \iff (\neg Q \implies \neg P).$$

2. State the negation of the following statements (fully simplified!):
   
   (a) $\exists M > 0$, s.t. $\forall x \in \mathbb{R}, f(x) < M$
   
   (b) $\exists L \in \mathbb{R}$, s.t. $\forall \epsilon > 0, \exists M > 0$, s.t. $\forall x \in \mathbb{R}$,
       $x > M \implies |f(x) - L| < \epsilon$

3. Suppose $A, B, C \subseteq X$ where $X$ is some universal set. Prove the following set identities:

   (a) $A \cap B = (A^c \cup B^c)^c$
   
   (b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   
   (c) $(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$

4. A positive integer $p$ is called a prime number if it is greater than 1 and the only whole divisors are 1 and $p$ itself. It can be easily checked that, up to the first ten such numbers, the set $P$ of prime numbers is given by

   $$P = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, \ldots\}.$$ 

   Prove that $\sqrt{p}$ is an irrational number for any prime number $p$. (Hint: You may use the fact that if $n^2$ is divisible by $p$ for any prime, then $n$ is divisible by $p$.)

5. Consider the following statement

   $$P = \text{“Every prime number greater than 2 is an odd number.”}$$

   (a) Restate $P$ (in words) in the form “If _____ and _____ then _____”.
   
   (b) State the contrapositive of $P$. Is this statement true? (Either prove it or provide a counter example.)
(c) State the converse of $P$. Is this statement true? (Either prove it or provide a counter example.)

6. Prove that the sequence $\{a_1, a_2, a_3, \ldots \}$ defined by

$$a_1 = 15, \quad a_{n+1} = 1 + \sqrt{1 + a_n}, \quad n \geq 1$$

has the following properties:

(a) $a_n > 3$ for all $n \geq 1$; and
(b) $a_{n+1} < a_n$ for all $n \geq 1$.

**Honors Question** In addition to being able to write correct proofs, it is also important to be able to identify false proofs and why they fail. For example, consider the following:

**Claim:** $\sqrt{4}$ is an irrational number

**False Proof:** Suppose otherwise. This implies that $\sqrt{4} = m/n$ where $m, n \in \mathbb{Z}$ and $m$ and $n$ may be selected to have no common factors. It follows that $4n^2 = m^2$. We therefore have that $m^2$, and therefore $m$, is divisible by 4. It follows that $m = 4k$ for some $k \in \mathbb{Z}$. We therefore have that $4n^2 = (4k)^2 \implies 4n^2 = 16k^2 \implies n^2 = 4k^2$. It follows that $n^2$, and therefore $n$ itself, is divisible by 4. This, however, contradicts the assumption that $m$ and $n$ have no common factors. It follows that our assumption was incorrect, and therefore that $\sqrt{4}$ is an irrational number.

This result, of course, is not correct, since we know that $\sqrt{4} = 2$, which is clearly a rational number. Identify the mistake in the above argument and explain why it is false.