Stability of Chemical Reaction Networks

Matthew Douglas Johnston
University of Waterloo

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1 Background

- Chemical Reactions
- Chemical Reaction Models
- Stoichiometric Compatibility Classes
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2 HJF Systems
- Reaction Graphs
- Deficiency Zero
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   - Chemical Reaction Models
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2 HJF Systems
   - Reaction Graphs
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3 Reducible Systems
   - Uncharacterized systems
   - Reducible Systems
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   - Reaction Graphs
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3 Reducible Systems
   - Uncharacterized systems
   - Reducible Systems
In each elementary reaction, a set of reactants reacts at a given rate to form a product, e.g.

$$2H_2 + O_2 \rightarrow^{k} 2H_2O$$
In each elementary reaction, a set of reactants reacts at a given rate to form a product, e.g.

\[ 2\text{H}_2 + \text{O}_2 \overset{k}{\longrightarrow} 2\text{H}_2\text{O} \]

Species/Reactants
In each elementary reaction, a set of reactants reacts at a given rate to form a product, e.g.

\[ 2H_2 + O_2 \xrightarrow{k} 2H_2O \]

Reactant Complex
In each elementary reaction, a set of reactants reacts at a given rate to form a product, e.g.

\[ 2\text{H}_2 + \text{O}_2 \xrightarrow{k} 2\text{H}_2\text{O} \]

Product Complex
In each elementary reaction, a set of reactants reacts at a given rate to form a product, e.g.

$$2\text{H}_2 + \text{O}_2 \xrightarrow{k} 2\text{H}_2\text{O}$$

**Reaction Constant**
In each elementary reaction, a set of reactants reacts at a given rate to form a product, e.g.

\[ 2\text{H}_2 + \text{O}_2 \rightarrow^k 2\text{H}_2\text{O} \]

Chemical kinetics is the study of the rates/dynamics resulting from such reactions.
In each elementary reaction, a set of reactants reacts at a given rate to form a product, e.g.

\[ 2\text{H}_2 + \text{O}_2 \xrightarrow{k} 2\text{H}_2\text{O} \]

Chemical kinetics is the study of the rates/dynamics resulting from such reactions.

To build a mathematical model, we need to make physical assumptions, e.g.

- Uniform distribution (well-mixed);
- Temperature and volume are constant;
- Law of mass action applies.
There are two standard ways to index reaction systems: according to reactions and according to complexes.
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Reactions

\[ \mathcal{A}_1 \rightarrow \mathcal{A}_2 \]
\[ \mathcal{A}_2 \rightarrow \mathcal{A}_3 + \mathcal{A}_4 \]
\[ \mathcal{A}_3 + \mathcal{A}_4 \rightarrow \mathcal{A}_1 \]
There are two standard ways to index reaction systems: according to reactions and according to complexes.

\[ \mathcal{A}_1 \xrightarrow{k_1} \mathcal{A}_2 \]
\[ \mathcal{A}_2 \xrightarrow{k_2} \mathcal{A}_3 + \mathcal{A}_4 \]
\[ \mathcal{A}_3 + \mathcal{A}_4 \xrightarrow{k_3} \mathcal{A}_1 \]
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\[
\begin{align*}
\mathcal{A}_1 & \overset{k_1}{\longrightarrow} \mathcal{A}_2 \\
\mathcal{A}_2 & \overset{k_2}{\longrightarrow} \mathcal{A}_3 + \mathcal{A}_4 \\
\mathcal{A}_3 + \mathcal{A}_4 & \overset{k_3}{\longrightarrow} \mathcal{A}_1 \\
C_p^-(i) & \overset{k_i}{\longrightarrow} C_p^+(i)
\end{align*}
\]
There are two standard ways to index reaction systems: according to reactions and according to complexes.

\[ A_1 \xrightarrow{k_1} A_2 \]
\[ A_2 \xrightarrow{k_2} A_3 + A_4 \]
\[ A_3 + A_4 \xrightarrow{k_3} A_1 \]

\[ C_p^-(i) \xrightarrow{k_i} C_p^+(i) \]

\[ A_1 \rightarrow A_2 \]
\[ A_3 + A_4 \]

\[ \text{Complexes} \]

\[ \text{Reactions} \]
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### Reactions

\[
\begin{align*}
\mathcal{A}_1 & \xrightarrow{k_1} \mathcal{A}_2 \\
\mathcal{A}_2 & \xrightarrow{k_2} \mathcal{A}_3 + \mathcal{A}_4 \\
\mathcal{A}_3 + \mathcal{A}_4 & \xrightarrow{k_3} \mathcal{A}_1 \\
\mathcal{C}_p^-(i) & \xrightarrow{k_i} \mathcal{C}_p^+(i)
\end{align*}
\]

### Complexes

\[
\begin{align*}
\mathcal{A}_1 \xrightarrow{k(1,2)} & \mathcal{A}_2 \\
\mathcal{A}_1 \xleftarrow{k(3,1)} & \mathcal{A}_2 \\
\text{Net reaction:} & \mathcal{A}_3 + \mathcal{A}_4
\end{align*}
\]
There are two standard ways to index reaction systems: according to reactions and according to complexes.

**Reactions**

\[
\begin{align*}
A_1 & \xrightarrow{k_1} A_2 \\
A_2 & \xrightarrow{k_2} A_3 + A_4 \\
A_3 + A_4 & \xrightarrow{k_3} A_1 \\
C_p^-(i) & \xrightarrow{k_i} C_p^+(i)
\end{align*}
\]

**Complexes**

\[
\begin{align*}
A_1 & \xrightarrow{k(1,2)} A_2 \\
& \xleftarrow{k(3,1)} \xrightarrow{k(2,3)} A_3 + A_4 \\
C_i & \xrightarrow{k(i,j)} C_j
\end{align*}
\]
For the general system

\[ C_{p^-}(i) \xrightarrow{k_i} C_{p^+}(i), \quad i = 1, \ldots, r \]

we have the governing differential equations

\[
\dot{x} = \sum_{i=1}^{r} k_i (z_{p^+(i)} - z_{p^-(i)}) x^{z_{p^-(i)}}.
\]
For the general system

$$C_{p^-}(i) \xrightarrow{k_i} C_{p^+}(i), \quad i = 1, \ldots, r$$

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We have the following important components:

- we sum over $r$ reactions,
For the general system

\[ \mathcal{C}_{p^{-}(i)} \xrightarrow{k_{i}} \mathcal{C}_{p^{+}(i)}, \ i = 1, \ldots, r \]

we have the governing differential equations

\[ \dot{x} = \sum_{i=1}^{r} k_{i}(z_{p^{+}(i)} - z_{p^{-}(i)})x^{z_{p^{-}(i)}}. \]

We have the following important components:

- we sum over \( r \) reactions,
- \( k_{i} \) is the reaction rate,
For the general system

\[ C_{p-}(i) \xrightarrow{k_i} C_{p+}(i), \quad i = 1, \ldots, r \]

we have the governing differential equations

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\dot{x} = \sum_{i=1}^{r} k_i (z_{p+}(i) - z_{p-}(i)) x^{z_{p-}(i)}.
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We have the following important components:

- we sum over \( r \) reactions,
- \( k_i \) is the reaction rate,
- \( (z_{p+}(i) - z_{p-}(i)) \) is the reaction vector,
For the general system

$$C_p^-(i) \xrightarrow{k_i} C_p^+(i), i = 1, \ldots, r$$

we have the governing differential equations

$$\dot{x} = \sum_{i=1}^{r} k_i (z_{p}^+(i) - z_{p}^-(i)) x^{z_{p}^-(i)}.$$

We have the following important components:

- we sum over $r$ reactions,
- $k_i$ is the reaction rate,
- $(z_{p}^+(i) - z_{p}^-(i))$ is the reaction vector, and
- $x^{z_{p}^-(i)} = \prod_{j=1}^{m} (x_j)^{z_{p}^-(i)j}$ is the mass-action term.
Consider the (reversible) system

\[ A_1 \xrightleftharpoons[k_2]{k_1} 2A_2. \]
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\[ \mathcal{A}_1 \xrightleftharpoons[k_2]{k_1} 2\mathcal{A}_2. \]

This has the governing dynamics

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = k_1 \begin{bmatrix}
-1 \\
2
\end{bmatrix} x_1 + k_2 \begin{bmatrix}
1 \\
-2
\end{bmatrix} x_2,
\]

where \( x_1 \) and \( x_2 \) are the concentrations of \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) respectively.
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Consider the (reversible) system

\[ A_1 \xleftrightarrow{k_1 \over k_2} 2A_2. \]

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What kind of properties does this system have?

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1 \\
-2
\end{bmatrix} x_2^2 \quad \text{\(= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \)}
\]

The (positive) equilibrium set is given by

\[
E = \left\{ x \in \mathbb{R}^2_+ \mid x_2 = \sqrt{\frac{k_1}{k_2} x_1} \right\}.
\]
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E = \left\{ x \in \mathbb{R}_+^2 \mid x_2 = \sqrt{\frac{k_1}{k_2} x_1} \right\}.
\]

For any \( k_1, k_2, x_1, x_2 \) we have \( f(x) \in S \) where

\[
S = \text{span} \left\{ \begin{bmatrix}
1 \\
-2
\end{bmatrix} \right\}.
\]
Figure: Previous system with $k_1 = k_2 = 1$. 
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The restriction of solutions is a general property:

**Theorem**

* Solutions $x(t)$ are restricted to **compatibility classes** such that

$$x(t) \in (S + x_0) \cap \mathbb{R}_+^m \quad \forall t \geq 0.$$
The restriction of solutions is a general property:

**Theorem**

*Solutions* \( x(t) \) *are restricted to compatibility classes such that*

\[
x(t) \in (S + x_0) \cap \mathbb{R}_+^m \quad \forall t \geq 0.
\]

We are interested in the following sense of equilibria:

**Definition**

A mass action system is said to have **locally stable dynamics** if there exists a unique positive equilibrium point within each compatibility class which is asymptotically stable.
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The reaction graph can be considered as a directed graph with complexes as nodes and reactions as directed connections between nodes.
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For example:

\[
\begin{align*}
  C_1 & \rightarrow C_2 \\
  C_3 & \rightarrow C_4 \\
  C_5 & \rightarrow C_6
\end{align*}
\]
The reaction graph can be considered as a directed graph with complexes as nodes and reactions as directed connections between nodes.

For example:

\[
\begin{align*}
C_1 & \xleftarrow{(2,1)} C_2 \\
(3,1) & \uparrow \\
C_3 & \xrightarrow{(3,4)} C_4 \\
C_5 & \xrightarrow{(5,6)} C_6
\end{align*}
\]
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For example:

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**Definition**

Connected components of a reaction graph are called **linkage classes**.
The reaction graph can be considered as a directed graph with complexes as nodes and reactions as directed connections between nodes.

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**Definition**

Connected components of a reaction graph are called **linkage classes**.

This example has two linkage classes.
The reaction graph can be considered as a directed graph with complexes as nodes and reactions as directed connections between nodes.

For example:

\[
\begin{align*}
C_1 & \xleftarrow{(2,1)} C_2 & C_3 & \xrightarrow{(3,4)} C_4 \\
(3,1) & \uparrow & (2,4) & \\
C_5 & \xrightarrow{(5,6)} C_6
\end{align*}
\]

**Definition**

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Definition

Connected components of a reaction graph are called linkage classes.

This example has two linkage classes.
We will need the following graph theoretical concepts:
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**Definition**

A chemical reaction network is **weakly reversible** if given a directed path from $C_i$ to $C_j$, there is a directed path from $C_j$ to $C_i$. 
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**Definition**

A chemical reaction network is **weakly reversible** if given a directed path from $C_i$ to $C_j$, there is a directed path from $C_j$ to $C_i$.

**Definition**

The **deficiency** of a system, $\delta$, is given by

$$\delta = n - s - \ell$$

where $n$ is the number of complexes, $s$ is the dimension of the stoichiometric space, and $\ell$ is the number of linkage classes.
Example

$A_1 \xrightleftharpoons[k_2]{k_1} 2A_2$

$3A_1 \xrightarrow{k_3} A_3$

$k_4 \quad k_5$

$A_1 + 2A_2$
### Example

\[
A_1 \xrightleftharpoons[k_2]{k_1} 2A_2 \\
3A_1 \xrightarrow{k_3} A_3 \\
A_1 + 2A_2
\]

The system is weakly reversible.
Example

\[ A_1 \overset{k_1}{\underset{k_2}{\rightleftharpoons}} 2A_2 \]

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\[ A_1 + 2A_2 \overset{k_4}{\underset{k_5}{\rightleftharpoons}} \]

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Example

\[ A_1 \xrightleftharpoons[k_2]{k_1} 2A_2 \quad \quad 3A_1 \xrightarrow{k_3} A_3 \]

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3A_1 & \xrightarrow[k_3]{k_3} A_3 \\
A_1 + 2A_2 & \xleftarrow[k_4]{k_4} \xrightarrow[k_5]{k_5}
\end{align*}
\]

The system is weakly reversible.

Compute the deficiency:
Example

\[ A_1 \xrightleftharpoons[k_2]{k_1} 2A_2 \quad 3A_1 \xrightarrow{k_3} A_3 \]

\[ A_1 + 2A_2 \]

The system is weakly reversible.

Compute the deficiency: \( n = 5 \),
The system is weakly reversible.

Compute the deficiency: \( n = 5, \ l = 2, \)
Example

\[ A_1 \xleftrightarrow{k_1} k_2 2A_2 \]

\[ 3A_1 \xrightarrow{k_3} A_3 \]

\[ A_1 + 2A_2 \]

The system is weakly reversible.

Compute the deficiency: \( n = 5, \ell = 2, \)
Example

\[
\begin{align*}
A_1 \quad & \underset{k_2}{\overset{k_1}{\rightleftharpoons}} \quad 2A_2 \\
3A_1 \quad & \overset{k_3}{\rightarrow} \quad A_3 \\
A_1 + 2A_2 \quad & \overset{k_4}{\leftarrow} \quad \overset{k_5}{\rightarrow}
\end{align*}
\]

The system is weakly reversible.

Compute the deficiency: \( n = 5, \ell = 2, \)
Example

\[ A_1 \xleftrightarrow{k_1}{k_2} 2A_2 \]

\[ 3A_1 \xrightarrow{k_3} A_3 \]

\[ k_4 \quad k_5 \]

\[ A_1 + 2A_2 \]

The system is weakly reversible.

Compute the deficiency: \( n = 5, \ell = 2, \) and \( s = 3. \)
The system is weakly reversible.

Compute the deficiency: \( n = 5 \), \( \ell = 2 \), and \( s = 3 \).

So \( \delta = n - s - \ell = 0 \), i.e. this is a zero deficiency system.
In 1972, Horn, Jackson and Feinberg presented a result relating the reaction graph and the stability of mass action systems [1, 2, 3].
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**Theorem (Zero Deficiency Theorem)**

*If a mass-action system is weakly reversible and has a deficiency of zero then it exhibits locally stable dynamics for all possible rate constants.*
In 1972, Horn, Jackson and Feinberg presented a result relating the reaction graph and the stability of mass action systems [1, 2, 3].

**Theorem (Zero Deficiency Theorem)**

*If a mass-action system is weakly reversible and has a deficiency of zero then it exhibits locally stable dynamics for all possible rate constants.*

The proof uses the following Lyapunov function:

$$L(x) = \sum_{i=1}^{m} x_i (\ln x_i - \ln x_i^* - 1) + x_i^*$$
### Example

<table>
<thead>
<tr>
<th>Reaction</th>
<th>Reaction</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1 \rightleftharpoons \frac{k_1}{k_2} 2A_2$</td>
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<tr>
<td>$A_1 + 2A_2$</td>
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</tr>
</tbody>
</table>
Example

\[ \begin{align*}
    A_1 & \xleftrightarrow{k_1}{k_2} 2A_2 \\
    3A_1 & \xrightarrow{k_3} A_3 \\
    k_4 & \swarrow \searrow k_5 \\
    A_1 + 2A_2
\end{align*} \]

Without even writing down the governing DEs, we know this has locally stable dynamics!
Example

Without even writing down the governing DEs, we know this has locally stable dynamics!

This result about the *dynamics* of the system depends solely on properties of the *graph* of the system!
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Are there locally stable systems lying outside of the scope of HJF systems?
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Example

\[ 2A_1 + A_2 \xrightarrow{k_1} A_1 + 3A_2 \]
\[ A_1 + 2A_2 \xrightarrow{k_2} 2A_1 \]
Are there locally stable systems lying outside of the scope of HJF systems?

**Example**

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2A_1 + A_2 \xrightarrow{k_1} A_1 + 3A_2
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Are there locally stable systems lying outside of the scope of HJF systems?

Example

\[ 2A_1 + A_2 \xrightarrow{k_1} A_1 + 3A_2 \]

\[ A_1 + 2A_2 \xrightarrow{k_2} 2A_1 \]

Despite this, even the Lyapunov function considered by HJF used directly fails to prove stability.
We need to consider an alternative approach if we are to prove the stability of this system.
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Consider the modified Lyapunov function

\[
\tilde{L}(x) = \sum_{i=1}^{m} c_i [x_i (\ln x_i - \ln x_i^* - 1) + x_i^*]
\]

where the \(c_i\) are positive constants.
We need to consider an alternative approach if we are to prove the stability of this system.

Consider the modified Lyapunov function

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\tilde{L}(x) = \sum_{i=1}^{m} c_i [x_i (\ln x_i - \ln x_i^* - 1) + x_i^*]
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where the \( c_i \) are positive constants.

If we choose \( c_1 = 2 \) and \( c_2 = 1 \) then

\[
\frac{d}{dt} \tilde{L}(x) = \nabla \tilde{L}(x) \cdot f(x) < 0
\]

for \( x \neq x^* \) in the compatibility class \( \implies \) asymptotic stability!
Using the modified Lyapunov function $\tilde{L}(x)$ expands the scope of systems with locally stable dynamics!
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Can we find a general class of systems for which the Lyapunov function $\tilde{L}(x)$ works?
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We have called our class of systems *reducible* systems since it amounts to reducing complex systems to HJF systems.
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Can we find a general class of systems for which the Lyapunov function $\tilde{L}(x)$ works? ... YES, we can!

We have called our class of systems reducible systems since it amounts to reducing complex systems to HJF systems.

Former UW graduate student Debbie MacLean also considered these problems.
Suppose there are $b_i > 0$ and $c_j > 0$, $i = 1, \ldots, r, j = 1, \ldots, m$, such that for every reactant complex $C_l$ there is a reaction (and vice-versa) such that

$$\frac{b_i c_j}{2} (z^j_{p^+ (i)} - z^j_{p^- (i)}) = (z^i_j - z^j_{p^- (i)}), \text{ for all } i, j.$$
Suppose there are $b_i > 0$ and $c_j > 0$, $i = 1, \ldots, r$, $j = 1, \ldots, m$, such that for every reactant complex $C_l$ there is a reaction (and vice-versa) such that

$$\frac{b_i c_j}{2} (z_{p+}^j(i) - z_{p-}^j(i)) = (z_{p-}^l(i) - z_{p+}^l(i)),$$  for all $i, j$.

This equation relates the graph of the original system to a “target” system with “similar” dynamics.
Suppose there are $b_i > 0$ and $c_j > 0$, $i = 1, \ldots, r$, $j = 1, \ldots, m$, such that for every reactant complex $C_l$ there is a reaction (and vice-versa) such that

$$\frac{b_i c_j}{2} (z_{p^+(i)}^j - z_{p^-(i)}^j) = (z_i^j - z_{p^-(i)}^j),$$

for all $i, j$.

Original system

This equation relates the graph of the original system to a “target” system with “similar” dynamics.
Suppose there are $b_i > 0$ and $c_j > 0$, $i = 1, \ldots, r$, $j = 1, \ldots, m$, such that for every reactant complex $C_l$ there is a reaction (and vice-versa) such that

$$\frac{b_i c_j}{2} (z^j_{p^+(i)} - z^j_{p^-(i)}) = (z^j_l - z^j_{p^-(i)})$$

for all $i,j$.

**Target system**

This equation relates the graph of the original system to a “target” system with “similar” dynamics.
Suppose there are $b_i > 0$ and $c_j > 0$, $i = 1, \ldots, r$, $j = 1, \ldots, m$, such that for every reactant complex $C_l$ there is a reaction (and vice-versa) such that

$$\frac{b_i c_j}{2}(z^j_{r^+ - (i)} - z^j_{r^- - (i)}) = (z^j_l - z^j_{p - (i)})$$

for all $i, j$.

This equation relates the graph of the original system to a “target” system with “similar” dynamics.

The target graph has no purely product complexes - the graph has been reduced.
Original system and “target” system have the same dynamics under the transformation

\[ y(t) = Tx(t) \]

where \( T = \text{diag}\{c_1, \ldots, c_m\}. \)
Original system and “target” system have the same dynamics under the transformation

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If the “target” system is weakly reversible and has zero deficiency, the system is said to be “reducible” and has locally stable dynamics for all sets of rate constants.
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If the “target” system is weakly reversible and has zero deficiency, the system is said to be “reducible” and has locally stable dynamics for all sets of rate constants.

Lyapunov function used is

\[ \tilde{L}(x) = \sum_{i=1}^{m} c_i [x_i (\ln x_i - \ln x_i^* - 1) + x_i^*] \]
Consider our previous system:

**Original System**

\[
2A_1 + A_2 \xrightleftharpoons[k_1]{k_2} A_1 + 3A_2 \\
A_1 + 2A_2 \xrightarrow{k_2} 2A_1
\]
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We need to pair
Consider our previous system:

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We need to pair reactant complexes
Consider our previous system:

\[ \begin{align*}
2A_1 + A_2 & \xrightarrow{k_1} A_1 + 3A_2 \\
A_1 + 2A_2 & \xrightarrow{k_2} 2A_1
\end{align*} \]

We need to pair reactant complexes and reactions.
Consider our previous system:

**Original System**

\[
2A_1 + A_2 \xrightleftharpoons[k_1]{\rightarrow} A_1 + 3A_2 \\
A_1 + 2A_2 \xrightarrow[k_2]{\rightarrow} 2A_1
\]

We need to pair reactant complexes and reactions such that every reaction occurs between reactant complexes.
Consider our previous system:

\[
\begin{align*}
\text{Original System} & \\
2A_1 + A_2 & \xrightarrow{k_1} A_1 + 3A_2 \\
A_1 + 2A_2 & \xrightarrow{k_2} 2A_1
\end{align*}
\]

\[
\begin{align*}
\text{Target System} & \\
2A_1 + A_2 & \overset{\tilde{k}_1}{\rightleftharpoons} A_1 + 2A_2.
\end{align*}
\]

We need to pair reactant complexes and reactions such that every reaction occurs between reactant complexes.

In fact, this is the only possible target graph.
We still need to check the algebraic conditions, i.e. find $b_1, b_2, c_1, c_2 > 0$ such that

\[
\frac{b_1}{2}\begin{bmatrix}
c_1 & 0 \\
0 & c_2
\end{bmatrix}\begin{bmatrix}
-1 \\
2
\end{bmatrix} = \begin{bmatrix}
-1 \\
1
\end{bmatrix}
\]

\[
\frac{b_2}{2}\begin{bmatrix}
c_1 & 0 \\
0 & c_2
\end{bmatrix}\begin{bmatrix}
1 \\
-2
\end{bmatrix} = \begin{bmatrix}
1 \\
-1
\end{bmatrix}.
\]
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\[
\begin{align*}
\frac{b_1}{2} & \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\
\frac{b_2}{2} & \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.
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\]

Original System
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\[
\frac{b_2}{2} \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.
\]

Target System
We still need to check the algebraic conditions, i.e. find $b_1, b_2, c_1, c_2 > 0$ such that

\[
\frac{b_1}{2} \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}
\]

\[
\frac{b_2}{2} \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.
\]

Can be found by inspection:

\[
b_1 = 1, b_2 = 1, c_1 = 2, c_2 = 1.
\]

The system exhibits locally stable dynamics!
There is still much work to be done in this area:
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- Determining whether large-scale systems are reducible is computationally cumbersome.

- Some systems have more than one valid “target” system.

- There remain systems with locally stable dynamics which are not reducible.
Acknowledgements

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