1. Evaluate the following improper integrals:

(a) $\int_0^1 \frac{1}{\sqrt{1-x}} \, dx$

**Solution:** The height of the rectangles shoot to infinity near $x = 1$ so we will integrate over the well-defined region $x = 0$ to $x = a$ and then take the limit as $a \to 1$ from the left. We have

$$
\int_0^1 \frac{1}{\sqrt{1-x}} \, dx = \lim_{a \to 0^+} \int_0^a \frac{1}{\sqrt{1-x}} \, dx \\
= \lim_{a \to 0^+} \left[ -\frac{3}{2} (1-x)^{2/3} \right]_0^a \\
= -\lim_{a \to 0^-} \frac{3}{2} (1-a)^{2/3} + \frac{3}{2} \\
= \frac{3}{2}.
$$

(b) $\int_0^\infty e^{-2x} \sin(3x) \, dx$

**Solution:** We need to take the integral from $x = 0$ to $x = a$ and then the limit as $a \to \infty$. Applying integration by parts, we have

$$
\int_0^\infty e^{-2x} \sin(3x) \, dx = \lim_{a \to \infty} \int_0^a e^{-2x} \sin(3x) \, dx \\
= \lim_{a \to \infty} -\frac{1}{13} \left[ 3e^{-2x} \cos(3x) + 2e^{-2x} \sin(3x) \right]_0^a \\
= -\frac{1}{13} \left[ 3e^{-2a} \cos(2a) + 2e^{-2a} \sin(3a) - 3 \right] \\
= \frac{3}{13}.
$$

(c) $\int_1^\infty \frac{1}{x \ln x} \, dx$
Solution: We need to make the substitution $u = \ln(x)$. This gives

$$\frac{du}{dx} = \frac{1}{x} \implies du = \frac{dx}{x}.$$  

The new bounds are $x = \infty \implies u = \infty$ and $x = 1 \implies u = 0$. This implies that

$$\int_1^\infty \frac{1}{x \ln(x)} \, dx = \int_0^\infty \frac{1}{u} \, du.$$

We notice neither bound is proper so that we need to take the integral from $u = a$ to $u = b$ and then the limit as $a$ approaches zero from the right and $b$ approaches infinity. We have

$$\int_0^\infty \frac{1}{u} \, dx = \lim_{a \to 0^+} \lim_{b \to \infty} \int_a^b \frac{1}{u} \, du = \lim_{a \to 0^+} \lim_{b \to \infty} \left[ \ln(u) \right]_a^b = \lim_{b \to \infty} \ln(b) - \lim_{a \to 0^-} \ln(a) = \infty.$$

(d) $\int_{-\infty}^{\infty} \frac{1}{1 + x^2} \, dx$

Solution: We need to take the integral from $x = a$ to $x = b$ and then let $a$ approach negative infinite and $b$ approach positive infinity. We have

$$\int_{-\infty}^{\infty} \frac{1}{1 + x^2} \, dx = \lim_{a \to -\infty} \lim_{b \to \infty} \int_a^b \frac{1}{1 + x^2} \, dx = \lim_{a \to -\infty} \lim_{b \to \infty} \left[ \arctan(x) \right]_a^b = \lim_{b \to \infty} \arctan(b) - \lim_{a \to -\infty} \arctan(a) = \frac{\pi}{2} - \left( -\frac{\pi}{2} \right) = \pi.$$

2. Evaluate the following integral:

$$\int_0^{\pi/4} \frac{\cos(t)t - \sin(t)}{t^2} \, dt.$$

(Hint: Find a function which gives this by the quotient rule!)
Solution: The form for the quotient rule is
\[
\frac{d}{dt} \left[ \frac{f(t)}{g(t)} \right] = \frac{f'(t)g(t) - f(t)g'(t)}{g(t)^2}.
\]
It should be fairly obvious that the required function is
\[
h(t) = \frac{f(t)}{g(t)} = \frac{\sin(t)}{t}.
\]
The height of the original integral becomes unbounded near \( t = 0 \) so that we need to take the integral from \( t = a \) to \( t = \pi/4 \) and then the limit as \( a \) goes to zero from the right-hand side. This gives
\[
\int_0^{\pi/4} \frac{\cos(t)t - \sin(t)}{t^2} \, dt = \lim_{a \to 0^+} \int_a^{\pi/4} \frac{\cos(t)t - \sin(t)}{t^2} \, dt
\]
\[
= \lim_{a \to 0^+} \left[ \frac{\sin(t)}{t} \right]_a^{\pi/4}
\]
\[
= \frac{\sin(\pi/4)}{\pi/4} - \lim_{a \to 0^+} \frac{\sin(a)}{a}
\]
\[
= \frac{1}{\sqrt{2}} - 1
\]
\[
= \frac{2\sqrt{2}}{\pi} - 1.
\]

3. Find the volume of the solid produced by the region bound by \( y = x^2 \) and \( y = \sqrt{x} \) in the first quadrant rotated around the \( x \)-axis. Determine this volume using both the “washer” and “shell” methods.

Solution: We can see that the region in the \( xy \)-plane we are interested in runs from \( 0 \) to \( 1 \) in both the \( x \) and \( y \) directions. Using the washer method, we integrate from \( x = 0 \) to \( x = 1 \) the shape given by the upper function rotated about the \( x \)-axis \( (\pi f(x)^2) \) minus the lower function
rotated about the same axis \((\pi g(x)^2)\). All told, we have

\[
\text{Volume} = \pi \int_0^1 \left[ (\sqrt{x^2} - (x^2)^2) \right] dx
\]

\[
= \pi \int_0^1 [x - x^4] dx
\]

\[
= \pi \left[ \frac{x^2}{2} - \frac{x^5}{5} \right]_0^1
\]

\[
= \pi \left[ \frac{1}{2} - \frac{1}{5} \right]
\]

\[
= \frac{3\pi}{10}.
\]

Using the shell method, we have to switch the variable dependences. We have to integrate from \(y = 0\) to \(y = 1\) the shape given by the upper function (in \(x\!\!) \) minus the lower function (in \(x\!\!) \) rotated about the \(x\) axis. Switching the variable dependences, we have \(y = \sqrt{x} \implies x = y^2\) and \(y = x^2 \implies x = \sqrt{y}\). It follows that we need

\[
\text{Volume} = 2\pi \int_0^1 y (\sqrt{y} - y^2) dy
\]

\[
= 2\pi \int_0^1 \left( y^{3/2} - y^3 \right) dy
\]

\[
= 2\pi \left[ \frac{2}{5} y^{5/2} - \frac{y^4}{4} \right]_0^1
\]

\[
= 2\pi \left[ \frac{2}{5} - \frac{1}{4} \right]
\]

\[
= \frac{3\pi}{10}.
\]

4. Find the volume of the solid produced by the region bound by \(y = x^2\) and \(y = (1/2)x^2 + 2\) rotated about the \(y\)-axis. You may use any method you like.

**Solution:** The curves intersect when \(x^2 = (1/2)x^2 + 2\) which gives

\[
\frac{1}{2}x^2 - 2 = 0 \implies x^2 - 4 = 0 \implies (x - 2)(x + 2) = 0.
\]
This has solutions $x = 2$ and $x = -2$. The easiest method for the integral will be the shell method, which rotates around the y-axis naturally. We have

$$\text{Volume} = 2\pi \int_0^2 x \left( \frac{1}{2} x^2 + 2 - x^2 \right) \, dx$$

$$= \pi \int_0^2 (4x - x^3) \, dx$$

$$= \pi \left[ 2x^2 - \frac{x^4}{4} \right]_0$$

$$= \pi [2(4) - 4]$$

$$= 4\pi.$$

5. Consider a spherical fishbowl with a radius of 10 cm. The top of the fishbowl is cut-off at a height of 15 cm to allow the bowl to be filled. Find the amount of water it would take to fill the fishbowl to the top.

**Solution:** We set this up as a sphere centred at $(0,0)$ with radius 10 (i.e. $x^2 + y^2 = 100$). We will take slices in the positive $x$ direction (from $x = 0$ to $x = \sqrt{100 - y^2}$) of width $dy$ and swing them around the $y$-axis. The restriction that the top is cut off 5 cm from the top amounts to integrating these circular slices (of volume $\pi r^2 dy$) over the range $y = -10$ to $y = 5$. We have

$$\text{Volume} = \pi \int_{-10}^5 \left( \sqrt{100 - y^2} \right)^2 \, dy$$

$$= \pi \int_{-10}^5 (100 - y^2) \, dy$$

$$= \pi \left[ 100y - \frac{y^3}{3} \right]_{-10}^5$$

$$= \pi \left( 500 - \frac{125}{3} \right) - \left( -1000 + \frac{1000}{3} \right)$$

$$= \pi \left[ 1500 - \frac{1125}{3} \right]$$

$$= 1125\pi \text{ cm}^3.$$

6. (Section 7.1, Question 15) A cylindrical hole of radius $a$ is bored through a solid right-circular cone of height $h$ and base radius $b > a$. 
If the axis of the hole lies along that of the cone, find the volume of the remaining part of the cone.

**Solution:** A right-circular cone of height $h$ and radius $b$ can be given by the shape below the line going through $(0, h)$ and $(b, 0)$ rotated around the $y$-axis. This line is given by $y = -(h/b)x + h$ or $y = h(1 - x/b)$. We can consider drilling a hole of radius $a < b$ straight through the centre of the shape by using the shell method and integrating only from $x = a$ to $x = b$. We have

$$\text{Volume} = 2\pi \int_a^b x \left( -\left( \frac{h}{b} \right)x + h \right) dx$$

$$= 2\pi \left[ \frac{hx^2}{2} - \frac{hx^3}{3b} \right]_a^b$$

$$= 2\pi \left[ \left( \frac{hb^2}{2} - \frac{hb^3}{3b} \right) - \left( \frac{ha^2}{2} - \frac{ha^3}{3b} \right) \right]$$

$$= 2\pi \left[ \frac{hb^2}{6} - \frac{ha^2}{2} + \frac{ha^3}{3b} \right].$$