1. Determine the length of the arc

\[ f(x) = \frac{e^x + e^{-x}}{2} \]

from \( x = 0 \) to \( x = 1 \). (Hint: This can be done directly, but it can done faster using hyperbolic trigonometric identities.)

**Solution:** We can evaluate this directly using

\[ f'(x) = \frac{e^x - e^{-x}}{2} \]

to get

\[
\begin{align*}
\text{Arc length} &= \int_{0}^{1} \sqrt{1 + \left[ f'(x) \right]^2} \, dx \\
&= \int_{0}^{1} \sqrt{1 + \left( \frac{e^x - e^{-x}}{2} \right)^2} \, dx \\
&= \int_{0}^{1} \sqrt{1 + \frac{e^{2x}}{4} - \frac{1}{2} + \frac{e^{-2x}}{4}} \, dx \\
&= \int_{0}^{1} \left( \frac{e^{2x} + 1}{2} + \frac{e^{-2x}}{4} \right) \, dx \\
&= \int_{0}^{1} \left( \frac{e^{2x} + e^{-x}}{2} \right) \, dx \\
&= \int_{0}^{1} \frac{e^x + e^{-x}}{2} \, dx \\
&= \left[ \frac{e^x - e^{-x}}{2} \right]_0^1 = \frac{e^2 - 1}{2e}. \\
\end{align*}
\]

Alternatively, we could have noted that

\[ f(x) = \frac{e^x + e^{-x}}{2} = \cosh(x) \implies f'(x) = \sinh(x) \]
so that

\[
\text{Arc length} = \int_0^1 \sqrt{1 + \sinh^2(x)} \, dx = \int_0^1 \cosh(x) \, dx = [\sinh(x)]_0^1 = \sinh(1) - \sin(0) = \sinh(1).
\]

It can easily be verified that these numbers are identical.

2. Consider a circular cone of length \(h\) and an open-end radius of \(r\). Show that the surface area of this cone is

\[\pi r \sqrt{r^2 + h^2}.\]

**Solution:** The shape of a cone matching these specifications can be given by wrapping the line \(f(x) = (r/h)x\) defined from \(x = 0\) to \(x = h\) around the \(x\)-axis. The formula for the surface area of such a shape gives

\[
\text{Surface area} = 2\pi \int_0^h f(x) \sqrt{1 + [f'(x)]^2} \, dx = 2\pi \int_0^h \left(\frac{r}{h} x\right) \sqrt{1 + \left(\frac{r}{h}\right)^2} \, dx = 2\pi \left(\frac{r}{h}\right) \sqrt{1 + \left(\frac{r}{h}\right)^2} \int_0^h x \, dx = 2\pi \left(\frac{r}{h^2}\right) \sqrt{h^2 + r^2} \left[\frac{x^2}{2}\right]_0^h = \pi r \sqrt{h^2 + r^2}.
\]

3. Determine the surface area of the shape produced by wrapping the curve \(f(x) = 2\sqrt{x}\) from \(x = 0\) to \(x = 1\) around the \(x\)-axis.

**Solution:** We have

\[f'(x) = \frac{1}{\sqrt{x}}.\]
It follows by the surface area formula that

\[
\text{Surface area} = 2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2} \, dx
\]
\[
= 2\pi \int_0^1 (2\sqrt{x}) \sqrt{1 + \frac{1}{x}} \, dx
\]
\[
= 4\pi \int_0^1 \sqrt{x+1} \, dx
\]
\[
= 4\pi \left[ \frac{2}{3} (x+1)^{3/2} \right]_0^1
\]
\[
= \frac{8}{3} \pi \left[ 2^{3/2} - 1 \right] \approx 15.318.
\]

4. Consider the shape bound by \( f(x) = x^n \) and \( g(x) = x^m \) where \( m > n > 0 \) between \( x = 0 \) and \( x = 1 \).

(a) Prove that the centroid of this shape can be given by

\[
x^* = \frac{(n+1)(m+1)}{(n+2)(m+2)}, \quad y^* = \frac{(n+1)(m+1)}{(2n+1)(2m+1)}.
\]

Solution: Since \( x^n > x^m \) in this region, we have

\[
\text{Area} = \int_0^1 (x^n - x^m) \, dx
\]
\[
= \left[ \frac{x^{n+1}}{n+1} - \frac{x^{m+1}}{m+1} \right]_0^1
\]
\[
= \frac{1}{n+1} - \frac{1}{m+1} - \frac{1}{m-n}
\]
\[
= \frac{(n+1)(m+1)}{(n+1)(m+1)}
\]
\[ \int_a^b x(f(x) - g(x)) \, dx = \int_0^1 x(x^n - x^m) \, dx \\
= \int_0^1 (x^{n+1} - x^{m+1}) \, dx \\
= \left[ \frac{x^{n+2}}{n+2} - \frac{x^{m+2}}{m+2} \right]_0^1 \\
= \frac{1}{n+1} - \frac{1}{m+2} \\
= \frac{m-n}{(n+2)(m+2)} \]

and

\[ \frac{1}{2} \int_a^b (f(x)^2 - g(x)^2) \, dx = \frac{1}{2} \int_0^1 (x^{2n} - x^{2m}) \, dx \\
= \frac{1}{2} \left[ \frac{x^{2n+1}}{2n+1} - \frac{x^{2m+1}}{2m+1} \right]_0^1 \\
= \frac{1}{2} \left[ \frac{1}{2n+1} - \frac{1}{2m+1} \right] \\
= \frac{1}{2} \left( \frac{2m - 2n}{(2n+1)(2m+1)} \right) \\
= \frac{m-n}{(2n+1)(2m+1)}. \]

It follows directly that

\[ x^* = \frac{(n+1)(m+1)}{(n+2)(m+2)}, \quad y^* = \frac{(n+1)(m+1)}{(2n+1)(2m+1)}. \]

(b) Find the centroid of the shape bound by \( f(x) = \sqrt{x} \) and \( g(x) = x^2 \) between \( x = 0 \) and \( x = 1 \).

**Solution:** We can plug the values \( n = 1/2 \) and \( m = 2 \) into the result of part (a) to get

\[ x^* = y^* = \frac{9}{20}. \]

(c) What happens as \( n \to 0 \) and \( m \to \infty \)? To what shape does this region correspond?
Solution: We can take the limits sequentially. We have

\[
\lim_{m \to \infty} \lim_{n \to 0} x^* = \lim_{m \to \infty} \lim_{n \to 0} \frac{(n + 1)(m + 1)}{(n + 2)(m + 2)} = \lim_{m \to \infty} \frac{m + 1}{2m + 4} = \frac{1}{2}
\]

and

\[
\lim_{m \to \infty} \lim_{n \to 0} y^* = \lim_{m \to \infty} \lim_{n \to 0} \frac{(n + 1)(m + 1)}{(2n + 1)(2m + 1)} = \lim_{m \to \infty} \frac{m + 1}{2m + 1} = \frac{1}{2}.
\]

This should not come as a surprise. As we take \( n \to 0 \) and \( m \to \infty \) we can a region that looks closer and closer to the square with corners \((0, 0), (0, 1), (1, 0)\) and \((1, 1)\). It is obvious that the centroid of this shape is \((1/2, 1/2)\).

5. Approximate the area under the curve \( f(x) = \sqrt{4 - x^2} \) between \( x = 0 \) and \( x = 2 \) using the rectangular (right end-point), trapezoidal, and Simpson’s rules using \( n = 4 \). Which method gives the best estimate of the actual solution \([\text{Area}] = \pi\)?

Solution: For \( n = 4 \) and the interval length \( b - a = 2 \) we have \( h = \Delta x = (b - a)/n = 0.5 \). It follows that we have

\[
\begin{align*}
x_0 &= 0 \quad \Rightarrow \quad f(x_0) = 2 \\
x_1 &= 0.5 \quad \Rightarrow \quad f(x_1) = \sqrt{3.75} \\
x_2 &= 1 \quad \Rightarrow \quad f(x_2) = \sqrt{3} \\
x_3 &= 1.5 \quad \Rightarrow \quad f(x_3) = \sqrt{1.75} \\
x_4 &= 2 \quad \Rightarrow \quad f(x_4) = 0.
\end{align*}
\]
It follows that we have the following estimates of $\int_0^1 \sqrt{3 - x^2} \, dx$:

- Rectangular = \((0.5)[\sqrt{3.75} + \sqrt{3} + \sqrt{1.75} + (0)]\) 
  \approx 2.495709068
- Trapezoidal = \(\frac{(0.5)}{2} \left[ 2 + 2\sqrt{3.75} + 2\sqrt{3} + 2\sqrt{1.75} + (0) \right]\) 
  \approx 2.995709968
- Simpson’s = \(\frac{(0.5)}{3} \left[ 2 + 4\sqrt{3.75} + 2\sqrt{3} + 4\sqrt{1.75} + (0) \right]\) 
  \approx 3.083595155.

It can be clearly seen that the best estimate comes from the Simpson’s rule.

6. How many intervals \(n\) must be chosen in order to guarantee an estimate accurately to \(E = 0.01\) for the area under the curve of \(f(x) = \cos(x)e^{-x}\) between \(x = 0\) and \(x = 4\) using

(a) the trapezoidal rule, and
(b) Simpson’s rule.

**Solution:** We have

\[f''(x) = 2\sin(x)e^{-x} \implies |f''(x)| \leq 2(1)(1) = 2\]
on \(0 \leq x \leq 4\). It follows that \(M = \max_{0 \leq x \leq 4} |f''(x)| = 2\) so that

\[|T_n| = \frac{M(b-a)^3}{12n^2} \leq 0.01\]
gives us

\[\frac{(2)(4 - 0)^3}{12n^2} \leq 0.01 \implies n \geq \sqrt{\frac{(2)(4)^3}{12(0.01)}} \approx 32.7\]

It follows that we should choose 33 intervals for our estimate. We also have

\[f'''(x) = -4\cos(x)e^{-x} \implies |f'''(x)| \leq 4(1)(1) = 4\]
on \(0 \leq x \leq 4\). It follows that \(M = \max_{0 \leq x \leq 4} |f'''(x)| = 4\) so that

\[|S_n| = \frac{M(b-a)^5}{180n^4} \leq 0.01\]
gives us

\[
\frac{4(4 - 0)^5}{180n^4} \leq 0.01 \quad \implies \quad n \geq \sqrt[4]{\frac{4(4)^5}{180(0.01)}} \approx 6.9.
\]

It follows that we need only 7 intervals to guarantee an estimate within 0.01 of the true value.