1. Determine which of the following relationships can be written in the functional form $z = f(x, y)$. For those which can, find the functional form $z = f(x, y)$. For those which cannot, state a pair of points which contradict the functional requirements.

(a) $x + y + z^2 = 0$

Solution: We have $z^2 = -x - y$ which implies $z = \pm \sqrt{-x - y}$. This is not a functional form. For instance, the point $(x, y) = (-1, 0)$ can yield both the value $z = -1$ and $z = 1$. Since a single input point can yield two outputs, this is not a function.

(b) $y = 7x - 5z + 3$

Solution: We have

$$5z = 7x - y + 3 \implies z = \frac{7}{5}x - \frac{1}{5}y + \frac{3}{5}.$$  

It follows that this can be represented in the functional form $z = f(x, y)$ where $f(x, y) = \frac{7}{5}x - \frac{1}{5}y + \frac{3}{5}$.

(c) $xyz = 1$

Solution: We have

$$z = \frac{1}{xy}$$  

so that the required functional form $z = f(x, y)$ is $f(x, y) = 1/(xy)$. (Note that we do not require that the domain have no holes in it, just that each unique input corresponds to a unique output.)

(d) $\cos(xyz) = 1$

Solution: This equation can be satisfied if and only if

$$xyz = 2k\pi, \quad k \in \mathbb{Z}.$$  

In particular, if we select the value $k = 0$, we have that $xyz = 0$. Since this can be satisfied for any values $(0, 0, z)$ at all, this cannot be represented in a functional form $z = f(x, y)$. 

2. Sketch the domain of the following multivariate functions:

(a) \( f(x, y) = \ln(|xy|) \)

**Solution:** In order to be in the domain of \( \ln(\cdot) \) we need \(|xy| > 0\). Since the absolute value function returns strictly positive values for all interior values except 0, we have \(xy \neq 0\). It follows that the domain is

\[
D(f) = \{(x, y) \in \mathbb{R}^2 \mid x \neq 0 \text{ and } y \neq 0\}.
\]

(b) \( f(x, y) = \frac{1}{\sqrt{3x - y}} \)

**Solution:** We need the interior of a square root to be nonnegative. We also require that we cannot divide by zero, so we require \(3x - y > 0\). It follows that the domain is given by

\[
D(f) = \{(x, y) \in \mathbb{R}^2 \mid y < 3x\}.
\]

(c) \( f(x, y) = \sqrt{x^2 - y^2 - 1} \)

**Solution:** We need the interior of a square root to be nonnegative, so we have

\[
D(f) = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 \geq 1\}.
\]

(d) \( f(x, y) = \sqrt{4 - \frac{x}{y}} \)

**Solution:** We need the interior of a square root to be nonnegative, so we have \(4 - x/y \geq 0\) which implies \((4y - x)/y \geq 0\). This is true if either \(y > 0\) and \(y \geq (1/4)x\) or \(y < 0\) and \(y \leq (1/4)x\). It follows that

\[
D(f) = \{(x, y) \in \mathbb{R}^2 \mid y \geq (1/4)x \text{ and } y > 0, \text{ or } y \leq (1/4)x \text{ and } y < 0\}.
\]

(It is worth noting that it is not correct to just take \(4 - x/y \geq 0\), move things over to the right-hand side as \(4 \geq x/y\) and multiply across to get \(y \geq x\). This is because in the final multiplication step the sign of \(y\) is not determine and therefore requires further consideration (or two cases).)
3. Give a sketch of the level curves of the following multivariate functions. Be sure to label a few representative values of $C$.

(a) $f(x, y) = ye^{-x}$

**Solution:** We have

$$C = ye^{-x} \implies y = Ce^x.$$

It follows that the level curves are scalings of the exponential function (including negative values of $C$, which represent exponential functions beneath the $x$-axis).

(b) $f(x, y) = x^2 + 4y^2$
Solution: We have
\[ C = x^2 + 4y^2 \implies \left( \frac{x}{\sqrt{C}} \right)^2 + \left( \frac{y}{\sqrt{C}/2} \right)^2 = 1. \]

It follows that the level curves are ellipses stretching \( \sqrt{C} \) units in the \( x \) direction and \( \sqrt{C}/2 \) units in the \( y \) direction. The important geometrical consideration is that, for any value of \( C \), the ellipse is always stretched twice as far in the \( x \) direction than the \( y \). Also notice that \( C \geq 0 \).

(c) \( f(x, y) = y + yx^2 \) (Not submitted in final solutions)

Solution: We have
\[ C = y + yx^2 \implies y = \frac{C}{1 + x^2}. \]

This may or may not be a familiar shape. It has a peak (of \( C \)) at \( x = 0 \) and gradually spreads out toward zero in both directions as \( x \to \infty \) and \( x \to -\infty \). The function above represents scalings of this function fanning out in the \( y \) direction. This function should be familiar as the derivative of \( \arctan(x) \) (multiplied by a constant).

(d) \( f(x, y) = x + xy^2 \)

Solution: We have
\[ C = x + xy^2 \implies x = \frac{C}{1 + y^2}. \]

This is the inverse of the function in part (c). We have a peak in the \( x \) direction and spreading toward zero in the \( y \) direction, and as \( C \) varies this shape fans out in the \( x \) direction.

(e) \( f(x, y) = \sin(x) \cos(y) \)

Solution: We cannot simplify this any further, but we can analyze what happens. If we take \( C = \sin(x) \cos(y) \), we notice \( C \) cannot take any values below \(-1\) or above \( 1 \). Let’s consider a few representative cases. We can have \( \sin(x) \cos(y) = 1 \) if \( \sin(x) = 1 \) and \( \cos(y) = 1 \) or \( \sin(x) = -1 \) and \( \cos(y) = -1 \).
and $\cos(y) = 1$ or $\sin(x) = -1$ and $\cos(y) = -1$. This gives the set of points

$$x = \frac{\pi}{2} + 2n\pi,\ n \in \mathbb{Z}, \quad y = 2m\pi,\ m \in \mathbb{Z}$$

and

$$x = -\frac{\pi}{2} + 2n\pi,\ n \in \mathbb{Z}, \quad y = \pi + 2m\pi,\ m \in \mathbb{Z}.$$

We can have $\sin(x) \cos(y) = -1$ if $\sin(x) = 1$ and $\cos(y) = -1$ or $\sin(x) = -1$ and $\cos(y) = 1$. This gives the set of points

$$x = \frac{\pi}{2} + 2n\pi,\ n \in \mathbb{Z}, \quad y = \pi + 2m\pi,\ m \in \mathbb{Z}$$

and

$$x = \frac{\pi}{2} + 2n\pi,\ n \in \mathbb{Z}, \quad y = 2m\pi,\ m \in \mathbb{Z}.$$

We have can $\sin(x) \cos(y) = 0$ if $\sin(x) = 0$ or $\cos(y) = 0$. This gives the set of points

$$x = n\pi,\ n \in \mathbb{Z}, \quad \text{or} \quad y = \frac{\pi}{2} + m\pi,\ m \in \mathbb{Z}.$$

Figure 2: Contour plots of the multivariate functions in Question #3.
4. Determine whether the following multivariable functions have a limit at \((0,0)\). If they have a limit, determine its value. If they do not, find (at least) two paths along which different limits are attained at \((0,0)\).

(a) \(f(x, y) = \frac{x^2}{x^2 + y^2}\)

Solution: The level curves are given by

\[ C = \frac{x^2}{x^2 + y^2} \implies Cy^2 = (1 - C)x^2 \implies y = \pm \sqrt{\frac{C}{1 - C}}x. \]

In other words, the level curves are lines through \((0,0)\). It appears they take different values near \((0,0)\). To check, we approach \((0,0)\) along lines \(y = tx\), which gives

\[ \lim_{x \to 0} f(x, tx) = \lim_{x \to 0} \frac{x^2}{x^2 + (tx)^2} = \lim_{x \to 0} \frac{1}{1 + t^2} = \frac{1}{1 + t^2}. \]

It is clear that this can take different values at \((0,0)\) so that the limit of \(f(x, y)\) at \((0,0)\) does not exist.

(b) \(f(x, y) = \frac{xy^2}{x^2 + y^4}\)

Solution: The level curves are given by

\[ C = \frac{xy^2}{x^2 + y^4} \implies Cy^4 - xy^2 + Cx^2 = 0 \]

\[ \implies y^2 = \frac{x \pm \sqrt{x^2 - 4C^2x^2}}{2C} \]

\[ \implies y = \pm \sqrt{\frac{1 \pm \sqrt{1 - 4C^2}}{2C}}x. \]

It looks like the level curves are roots which take different values as they approach \((0,0)\). To check, we approach \((0,0)\) along curves \(y = t\sqrt{x}\) to get

\[ \lim_{x \to 0} f(x, t\sqrt{x}) = \lim_{x \to 0} \frac{x(t\sqrt{x})^2}{x^2 + (t\sqrt{x})^4} = \lim_{x \to 0} \frac{t^2x^2}{x^2 + t^4x^2} = \lim_{x \to 0} \frac{t^2}{1 + t^4}. \]
Since this can take different values depending on the values of \( t \), we have that the limit \( f(x, y) \) at \((0,0)\) does not exist.

(c) \( f(x, y) = \frac{x^2 y}{x^2 + y^2} \)

**Solution:** The level curves are given by

\[
C = \frac{x^2 y}{x^2 + y^2} \implies Cx^2 - yx^2 + Cy^2 = 0
\]

\[
\implies x^2(C - y) = -Cy^2 \implies x = \pm \sqrt{\frac{Cy^2}{y - C}}.
\]

This is a complicated form. We might notice, however, that we need the interior of the root to nonnegative. This implies that we need either

\[
C \geq 0, \text{ and } y - C > 0 \implies y > C \geq 0
\]

or

\[
C \leq 0, \text{ and } y - C < 0 \implies y < C \leq 0.
\]

In other words, we cannot approach \( y = 0 \) along a level curve unless \( C = 0 \). We might suspect, therefore, that only the single value \( C = 0 \) is attained in the limit of \( f(x, y) \) from any direction of \((x, y) \to (0,0)\). We check the definition to get

\[
\left| \frac{x^2 y}{x^2 + y^2} - 0 \right| \leq \left| \left( \frac{x^2 + y^2}{} \right) y \right| = |y| = \sqrt{y^2} \leq \sqrt{x^2 + y^2}.
\]

It follows that, as \( \sqrt{x^2 + y^2} \to 0 \), we have

\[
\frac{x^2 y}{x^2 + y^2} \to 0.
\]

By definition, we have

\[
\lim_{(x,y) \to (0,0)} \frac{x^2 y}{x^2 + y^2} = 0.
\]