University of Waterloo
Final Examination
SYDE 112
Fundamental Engineering Math 2

Instructor: Matthew Douglas Johnston
Date: Tuesday, July 31, 2012
Term: 1125
Number of exam pages: 10
(including cover page)

Section: 001
Time: 9:00 a.m.
Duration of exam: 2 hours, 30 minutes
Exam type: Closed Book

Formulas
- $\sin 2\alpha = 2\sin \alpha \cos \alpha$
- $\sin^2 \alpha = \frac{1 - \cos 2\alpha}{2}$
- $\cos^2 \alpha = \frac{1 + \cos 2\alpha}{2}$

Instructions
1. Write your name and ID number at the top of this page.
2. Answer the questions in the spaces provided, using the backs of pages for overflow or rough work.
3. No graphing calculators are permitted! (You should not, however, need a calculator.)
4. Show all your work required to obtain your answers.

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1. Evaluate the following integrals:

\[ \int \cos(x) \sqrt{\sin(x)} \, dx \]

(a) \[
= \int u^{\frac{1}{2}} \, du \\
= \frac{2}{3} u^{\frac{3}{2}} + C \\
= \frac{2}{3} [\sin(x)]^{\frac{3}{2}} + C
\]

\[ \int \frac{x-1}{\sqrt{x^2 - 2x - 3}} \, dx \] (Hint: Trigonometric substitution is not required.)

(b) \[
= \frac{1}{2} \int u^{\frac{1}{2}} \, du \\
= u^{\frac{1}{2}} + C \\
= (x^2 - 2x - 3)^{\frac{1}{2}} + C
\]

\[ \int (x+1)e^{-2x} \, dx \]

(c) \[
= \int x e^{-2x} \, dx + \int e^{-2x} \, dx \\
= -\frac{x e^{-2x}}{2} + \frac{1}{2} \int e^{-2x} \, dx + \int e^{-2x} \, dx \\
= -\frac{x e^{-2x}}{2} + \frac{3}{2} \int e^{-2x} \, dx \\
= -\frac{x e^{-2x}}{2} - \frac{3}{4} e^{-2x} + C
\]
\[
\begin{align*}
x &= -2 \cos(\theta x) \\
&\quad \left( \frac{\sin(\theta x)}{\sin(\theta)} \right) \\
\int_0^\infty \frac{e^{-x} \cos(\theta x)}{x} \, dx &= (\pi)_0 \\
\text{Evaluate } \delta(x) \text{ for the following function:}
\end{align*}
\]
3. Consider the shape produced by wrapping the arc \( f(x) = \ln(x) \) from \( x = 1 \) to \( x = 2 \) around the \( x \)-axis.

(a) Showing all of your steps, verify that the surface area of this shape can be given by

\[
\text{Surface area} = 2\pi \int_1^2 \sqrt{1 + x^2} \, dx.
\]

(Do not attempt to evaluate this integral!)

\[
SA = 2\pi \int_a^b x \sqrt{1 + \left( \frac{f'(x)}{x} \right)^2} \, dx
\]

\[
= 2\pi \int_1^2 \sqrt{1 + \left( \frac{1}{x} \right)^2} \, dx
\]

\[
= 2\pi \int_1^2 \sqrt{\frac{x^2 + 1}{x^2}} \, dx
\]

\[
= 2\pi \int_1^2 \sqrt{1 + x^2} \, dx
\]

\[
f(x) = \ln(x), \quad f'(x) = \frac{1}{x}
\]

(b) Set up the equation for estimating the surface area of this shape using the Trapezoidal rule and \( n = 4 \). (You do not need to evaluate the sum!)

Trapezoidal Rule: \( \int_a^b f(x) \, dx \approx \frac{h}{2} \left( f(a) + 2 \sum_{i=1}^{n-1} f(x_i) + f(b) \right) \)

\[
h = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4}
\]

\[
\int_a^b f(x) \, dx = (2\pi) \left( \frac{1}{4} \right) \left( \frac{1}{2} \right) \left[ \sqrt{1 + (1)^2} + 2 \sqrt{1 + (1.25)^2} + 2 \sqrt{1 + (1.75)^2} + 2 \sqrt{1 + (2)^2} \right]
\]

\[
\approx 11.37925999
\]

(c) Given that \( \frac{d^2}{dx^2} \left[ \sqrt{1 + x^2} \right] = \frac{1}{(1 + x^2)^{3/2}} \), determine an upper bound on the difference between the estimate of the integral produced in part (b) and the true value of the integral.

\[
|T_n| \leq \frac{M(b-a)^3}{12n^2}, \quad M = \max_{a \leq x \leq b} |f''(x)|.
\]

\[
M = \max_{1 \leq x \leq 2} \left| \frac{1}{(1 + x^2)^{3/2}} \right| = \frac{1}{(1 + (2)^2)^{3/2}} = \frac{1}{2^{3/2}}
\]

\[
\Rightarrow |T_n| \leq \frac{\left( \frac{2\pi}{2^{3/2}} \right) (2 - 1)^3}{12(4)^2} \leq \frac{2\pi}{2^{3/2}(12)(16)} = \frac{\pi}{2^{3/2}(96)} = 0.011570007
\]
4. Consider the multivariate function \( f(x, y) = \frac{2x}{y - x^2} \).

(a) State the domain of \( f(x, y) \).

(b) Sketch a contour plot of \( f(x, y) \) with the level curves corresponding to \( x = -2, -1, 0, 1, 2 \) labelled. Indicate any breaks in the domain with dotted lines.

(c) Consider travelling along a path corresponding to \( y = 4 \). Based on the contour plot, what happens as we cross the point \( (x, y) = (-2, 4) \) from the left to the right.

(d) Based on the contour plot, conjecture as to whether \( \lim_{(x,y) \to (0,0)} f(x,y) \) exists or not. If it exists, state its value. If it does not exist, state your reasoning. (You do not need to prove your conjecture but you do need to justify your reasoning.)

(e) Considering \( f(x,y) \) does not exist, because level curves converge at \( (0,0) \).
(For this page, we are still considering the multivariate function \( f(x, y) = \frac{2x}{y-x^2} \)).

\[ \frac{\partial f}{\partial x} = \frac{(y-x^2) - (-4x^2)}{(y-x^2)^2} = \frac{2y + 2x^2}{(y-x^2)^2} = \frac{y + x^2}{(y-x^2)^2} \]

\[ \frac{\partial f}{\partial y} = \frac{2x}{(y-x^2)^2} \]

\[ \begin{align*}
T(x, y) &= f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b) \\
f_x(0, 1) &= 0 \quad f_y(0, 1) = 0 \\
f_y(0, 1) &= 2 \\
\Rightarrow \quad T(x, y) &= 0 + 2(x-0) + 0(y-1) = 2x
\end{align*} \]

\[ \begin{align*}
\text{Directional derivative} &= \frac{1}{\sqrt{u_1^2 + u_2^2}} \nabla f(a, b) \cdot \hat{u} \\
&= \frac{1}{\sqrt{3^2 + 4^2}} (2, 0) \cdot (3, 4) = \frac{6}{5}
\end{align*} \]

\[ \begin{align*}
\text{Direction of steepest ascent is the direction of the gradient} \\
\Rightarrow \quad \nabla f(0, 1) &= (2, 0), \\
The magnitude is &\quad \sqrt{f_x^2 + f_y^2} \\
&= \sqrt{2^2 + 0^2} = 2.
\end{align*} \]
6. (a) Find and classify the critical points of \( f(x, y) = x^4 + 4xy + 2y^2 \).

\[
\begin{align*}
A &= 3x^2 + 4y = 12, \\
B &= 6x + 4y - 12 = 0, \\
C &= 2x^2 + 4y - 6 = 0.
\end{align*}
\]

Setting \( \frac{\partial f}{\partial x} = 0 \) and \( \frac{\partial f}{\partial y} = 0 \) gives:

\[
\begin{align*}
x &= 0, \\
y &= 0.
\end{align*}
\]

(b) Use the method of Lagrange multipliers to determine the maximal and minimal values of \( f(x, y) = x^4 + 4xy + 2y^2 \) subject to the constraint \( x^2 + y^2 = 1 \).

Let \( L(x, \lambda) = (x^4 + 4xy + 2y^2) + \lambda(x^2 + y^2 - 1) \).

\[
\begin{align*}
L_x &= -2x + 2y = 0, \\
L_y &= 1 + 2x = 0, \\
L_\lambda &= x^2 + y^2 = 0.
\end{align*}
\]

Solving the system of equations gives:

\[
\begin{align*}
(x_1, y_1) &= (0, 0), \\
(\lambda_1) &= 1, \\
(\lambda_2) &= 0.
\end{align*}
\]

\[
\begin{align*}
L_x(0, 0, 1) &= -2 \cdot 0 + 2 \cdot 0 = 0, \\
L_y(0, 0, 1) &= 1 + 2 \cdot 0 = 0, \\
L_\lambda(0, 0, 1) &= 0^2 + 0^2 = 0.
\end{align*}
\]

Therefore, \( f(0, 0, 1) = 0 \).
5. Consider the plane \( f(x, y) = x - y \).

(a) Determine the point on \( f(x, y) \) which is closest to the point \( (2, 0, -1) \).

\[
(x, y, z) = (a_t b_t, f_t(a_t b_t) + t(f_x(a_t b_t) + f_y(a_t b_t) - 1)
\]

\[
\begin{align*}
  f_x(a_t b_t) &= a - b \\
  f_y(a_t b_t) &= 1 \\
  f_x(a_t b_t) &= -1
\end{align*}
\]

\[
\Rightarrow (2, 0, -1) = (a_t b_t, (a - b) + t(1, 1, -1))
\]

\[
\Rightarrow a + t = 2 \\
  b - t = 0 \\
  a - b - t = -1
\]

\[
\Rightarrow (2 - t) - t - t = 1 \\
\Rightarrow -3t = -1 \\
  t = \frac{1}{3}
\]

\[
\begin{align*}
  a &= \frac{5}{3}, \\
  b &= \frac{1}{3}
\end{align*}
\]

(b) What is the maximal value of \( f(x, y) \) restricted to

\[
\text{(i) the interior of the unit square } S = \{(x, y) \in \mathbb{R}^2 \mid -1 \leq x \leq 1, -1 \leq y \leq 1\}?
\]

(Hint: There are no critical points.)

Check corner points:

\[
\begin{align*}
  f(-1, 1) &= 0 \\
  f(-1, -1) &= -2 \leq \min \\
  f(1, 1) &= 2 \leq \max \\
  f(1, -1) &= 0
\end{align*}
\]

\[
\text{(ii) the interior of the unit ball } B = \{(x, y) \in \mathbb{R}^2 \mid \sqrt{x^2 + y^2} \leq 1\}?
\]

\[
\begin{align*}
  x &= \cos \theta \\
  y &= \sin \theta
\end{align*}
\]

\[
\begin{align*}
  f(\theta) &= \cos \theta - \sin \theta, \quad \theta \in [0, 2\pi) \\
  f'(\theta) &= -\sin \theta - \cos \theta = 0
\end{align*}
\]

\[
\Rightarrow \tan \theta = -1
\]

\[
\Rightarrow \theta = \frac{3\pi}{4}, \quad \frac{7\pi}{4}
\]

\[
\begin{align*}
  \theta = \frac{3\pi}{4} \Rightarrow x &= \cos \left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2} \\
  y &= \sin \left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}
\end{align*}
\]

\[
\begin{align*}
  f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) &= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2} \leq \min \\
  \theta = \frac{3\pi}{4} \Rightarrow x &= \frac{\sqrt{2}}{2} \\
  y &= -\frac{\sqrt{2}}{2}
\end{align*}
\]

\[
\begin{align*}
  f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) &= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = 2 \sqrt{2} \leq \max
\end{align*}
\]
7. Consider the variable dependences \( u = x + y, v = x - y, z = x \). Determine expressions for \( \partial u / \partial x \) and \( \partial u / \partial y \).

8. (BONUS) Consider the multivariate function \( f(u, v) \) where \( u = x^2 - y^2 \) and \( v = 2xy \).

Show that

\[ \frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} = 0 \]