Consider the problem of determining the area below the curve $f(x)$ bound between two points $a$ and $b$.

For simple geometrical functions, we can easily determine this based on intuition. For a constant function $f(x) = c$, we know that the area below the curve between any two points is given by the length $(b-a)$ times the width $(c)$ so that

$$f(x) = c \implies \text{area bound between } a \text{ and } b \text{ is } (b-a)c.$$ 

Now consider the function $f(x) = cx$ and, for simplicity, suppose we are interested in the area bound between $x = 0$ and $x = 1$. This area is triangular, so we can easily compute the area as one half base $(1)$ times height $(c)$ so that

$$f(x) = cx \implies \text{area bound between } 0 \text{ and } 1 \text{ is } \frac{c}{2}.$$ 

![Figure 1: Areas under the curve for $f(x) = c$ (bound between $a$ and $b$) and $f(x) = cx$ (bound between 0 and 1).](image)

This process seems simple enough. We know the area of basic geometric shapes (e.g. squares, rectangles, triangles, parallelograms, circles, etc.) so
we might suspect that this knowledge will allow us to compute general areas. This intuition, however, is quickly shown to be lacking. If we consider even the simple parabola \( f(x) = x^2 \) bound between \( x = 0 \) and \( x = 1 \), we see that the area below the curve does not correspond to any easily computed geometric shape (see Figure 2).

![Figure 2: The area under the curve \( f(x) = x^2 \) bound between 0 and 1 does not correspond to any well-known, easily-computable geometric shape.](image)

It is certainly not satisfactory to throw our hands in the air and give up. Suppose we were in the carpeting business and had been asked how much carpet would be needed to carpet a room with dimensions given by that in Figure 2. The customer would not be very happy if we told him to rebuild his room in the form of a rectangle and get back to us!

In this situation, we would like to take what we know about easily-computable shapes to build an approximation of the actual area. We might as well start with the easiest shape we know, the rectangle. Imagine covering the area given in Figure 2 by two rectangles of equal width (1/2). If we imagine the height of those rectangles being given by where the right side hits \( x^2 \), and where the left side hits \( x^2 \), respectively, then we arrive at Figure 3. We can easily see that the area in Figure 3(a) overestimates the area, and the area in (b) underestimates it, so that under the curve is \( x^2 \) lies between the areas of the rectangles given in this picture! That is to say, we have

\[
(0) + \left( \frac{1}{4} \right) \left( \frac{1}{2} \right) < \text{Area under } x^2 < \left( \frac{1}{4} \right) \left( \frac{1}{2} \right) + 1 \left( \frac{1}{2} \right)
\]

\[
\frac{1}{8} < \text{Area under } x^2 < \frac{5}{8}.
\]
Figure 3: The area under the curve of \( f(x) = x^2 \) estimated using two rectangles. The height of the rectangles is given by the right point of the function in (a) and the left point in (b). Notice that \( f(0) = 0 \) so that the first rectangle in (b) has zero height.

This is a pretty big spread. We might wonder if we can do better. Of course we can! Imagine cutting estimating the area with four rectangles instead of two. Now we have the improved estimates for the area given in Figure 4, which computes to

\[
0 + \frac{1}{64} + \frac{1}{16} + \frac{9}{64} < \text{Area under } x^2 < \frac{1}{64} + \frac{1}{16} + \frac{9}{64} + \frac{1}{4}
\]

\[
\frac{7}{32} < \text{Area under } x^2 < \frac{15}{32}.
\]

This is a significantly better estimate! The difference between the upper and lower bound has been cut in half, from \( 1/2 \) to \( 1/4 \). We might wonder, based on this, how good of an estimate we can actually get by taking a greater and greater number of rectangles. The answer is that we can get as good of an estimate as we like! That is to say, if we want to estimate the area under \( x^2 \) to the fifth decimal place, there is some number of rectangles which will give it. If we want to estimate it to ten decimal places, there is some partition for that as well.

Let’s state this mathematically. In general, we want to estimate the area between \( x = a \) and \( x = b \) and we will take the width of each rectangle to be the same. This means that, if we have \( n \) rectangles, each one has width

\[
\Delta x = \frac{b - a}{n}.
\]

The height of each rectangle is given by some point on the curve \( f(x_i^*) \) where the \( x_i^* \) is taken to lie somewhere on the base of the rectangle. A common
Figure 4: The area under the curve of \( f(x) = x^2 \) estimated using four rectangles. The height of the rectangles is given by the right point of the function in (a) and the left point in (b). Notice that \( f(0) = 0 \) so that the first rectangle in (b) has zero height.

The choice is to take the \( x \)-value to be either the left-hand or right-hand endpoint, which correspond respectively to

\[
x_i^* = a + (i - 1)\Delta x \quad \text{(Left endpoint of \( i^{th} \) rectangle)}
\]

(2)

or

\[
x_i^* = a + i\Delta x \quad \text{(Right endpoint of \( i^{th} \) rectangle)}.
\]

(3)

The area of the \( i^{th} \) rectangle can be easily calculated as \([\text{base} \times \text{height}]\) to give

\[
f(x_i^*)\Delta x \quad \text{(Area of \( i^{th} \) rectangle)}
\]

from which it follows that the area under the curve \( f(x) \) bound between \( x = a \) and \( x = b \) is given by

\[
\sum_{i=1}^{n} f(x_i^*)\Delta x.
\]

It turns out that, for an appropriate choice of \( n \), this is actually a very good approximation of the area under a function, but we can do even better. We have some intuition that, if we take a “greater” and “greater” number of rectangles we are going to get a “better” and “better” estimate of the area. Well, what happens in the limit as the number of rectangles approaches infinity? Is it even sensible to try to evaluate this limit? After all, in the limit each rectangle has no width (i.e. zero area), but there are an infinite number of them.
It turns out that this limit is (usually) well-defined and accurately computes the area under the curve precisely! The Riemann sum is defined as
\[
\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x
\] (4)
where \(x_i^*\) is given by either (2) or (3), and \(\Delta x\) is given by (1). (We will connect this sum to a concept called integration, which is one of the foundational applications of calculus, in a future lecture.)

**Note:** Computing Riemann sums often requires the use of summation formulas. The most common forms encountered are
\[
\sum_{i=1}^{n} 1 = n
\] (5)
\[
\sum_{i=1}^{n} i = \frac{n(n+1)}{2}
\] (6)
\[
\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}
\] (7)
\[
\sum_{i=1}^{n} i^3 = \left( \frac{n(n+1)}{2} \right)^2
\] (8)

**Example 1:** Evaluate the Riemann sum for \(x^2\) bound between \(x = 0\) and \(x = 1\) using the right-endpoint (3).

**Solution:** We have
\[
\Delta x = \frac{b - a}{n} = \frac{1 - 0}{n} = \frac{1}{n}
\]
so that
\[
x_i^* = a + i \Delta x = 0 + i \left( \frac{1}{n} \right) = \frac{i}{n}
\]
This gives us

\[
\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x = \lim_{n \to \infty} \sum_{i=1}^{n} \left( \frac{i}{n} \right)^2 \left( \frac{1}{n} \right)
\]

\[
= \lim_{n \to \infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2
\]

\[
= \lim_{n \to \infty} \frac{n(n+1)(2n+1)}{6n^3}
\]

\[
= \lim_{n \to \infty} \frac{2n^3 + 3n^2 + n}{6n^3} = \frac{1}{3}.
\]

In other words, the area below $x^2$ between $x = 0$ and $x = 1$ is exactly $1/3$.

**Example 2:** Evaluate the Riemann sum for $x^3$ bound between $x = -1$ and $x = 1$ using the right-endpoint (3).

**Solution:** We have

\[
\Delta x = \frac{b - a}{n} = \frac{1 - (-1)}{n} = \frac{2}{n}
\]

so that

\[
x_i^* = a + i \Delta x = -1 + i \left( \frac{2}{n} \right) = -1 + \frac{2i}{n}.
\]

This gives us

\[
\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x
\]

\[
= \lim_{n \to \infty} \sum_{i=1}^{n} \left( 1 + \frac{2i}{n} \right)^3 \left( \frac{2}{n} \right)
\]

\[
= \lim_{n \to \infty} \frac{2}{n} \sum_{i=1}^{n} \left[ (-1)^3 + 6 \frac{i}{n} - 12 \frac{i^2}{n^2} + 8 \frac{i^3}{n^3} \right]
\]

\[
= \lim_{n \to \infty} \left[ -\frac{2}{n} \sum_{i=1}^{n} 1 + \frac{12}{n^2} \sum_{i=1}^{n} i - \frac{24}{n^3} \sum_{i=1}^{n} i^2 + \frac{16}{n^4} \sum_{i=1}^{n} i^3 \right]
\]

\[
= \lim_{n \to \infty} \left[ -2 + \frac{6(n+1)}{n} - \frac{24(2n^3 + 3n^2 + n)}{6n^3} + \frac{16(n^4 + 2n^3 + n^2)}{4n^4} \right]
\]

\[
= -2 + 6 - 8 + 4 = 0.
\]
In other words, there is no area under the curve $x^3$ bound between $x = -1$ and $x = 1$. Considering the graph of $x^3$, does this make sense? (Hint: rectangles drawn below the $x$-axis will produce a negative area!)