So far we have introduced two new concepts in this course: antidifferentiation and Riemann sums. It turns out that these quantities are related, but it is not immediately clear how. To more fully explore this connection, let us consider Riemann sums in more depth.

Our first step will be to give the Riemann sum a new name and notation. Rather than write out the sum in full detail each time we compute it, we will use the notation

$$\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x.$$  \hspace{1cm} (1)

This is called the definite integral and it is one of the foundational concepts of calculus. We will see later in this lecture that the definite integral can be defined in multiple ways, of which the Riemann sum formulation (1) is only one possibility. (The other definition will provide a very important connection with antidifferentiation!)

Consider the question of how the area under a function $f(x)$ changes as we move one of the bounds to the right or the left. For the sake of illustration, let’s keep the bound $a$ fixed and allow the bound $b$ to be given by the variable $x$. Given this set up, we will let

$$g(x) = \int_a^x f(x) \, dx$$

and

$$g(x + h) = \int_a^{x+h} f(x) \, dx.$$ 

That is to say, we let $g(x)$ correspond to the area under the curve from $a$ to $x$, and $g(x + h)$ to be the area under the curve from $a$ to $x + h$.

Now we ask the question: what is the difference between these two areas? It is easy to see that the amount of area added (or subtract, if below the axis) by extending the right bound from $x$ to $x + h$ is $g(x + h) - g(x)$. We can see that this area is very well approximated by the rectangle between $x$ and $x + h$ where we take the height to be $f(x)$. This rectangle has dimensions $h$ and $f(x)$ so that we have

$$g(x + h) - g(x) \approx hf(x)$$
or, written another way,

\[ \frac{g(x + h) - g(x)}{h} \approx f(x). \]

This is already starting to take a familiar form! But let’s make one more consideration. We can see that for well-behaved functions, this approximation becomes “better” and “better” as \( h \) becomes “smaller” and “smaller”, so that in the limit we have

\[ \lim_{h \to 0} \frac{g(x + h) - g(x)}{h} = f(x). \]

We immediately recognize the left-hand side as the derivative of \( g(x) \). In other words, we have \( g'(x) = f(x) \!\). What does this mean? The function \( g(x) \) corresponded to the area under the curve \( f(x) \) from the left-point \( a \) to the right-point \( x \). This tells us that if we take the function \( f(x) \), compute its area \( g(x) \), and take the derivative of that quantity, we recover the original function \( f(x) \). In other words, derivation undoes integration. This is called the First Fundamental Rule of Calculus.

**Theorem 0.1** (First Fundamental Theorem of Calculus). Suppose \( f \) is continuous on \([a,b]\). Then the function \( g \) defined by

\[ g(x) = \int_a^x f(t) \, dt \quad a \leq x \leq b \]

is continuous on \([a,b]\) and differentiable on \((a,b)\), and \( g'(x) = f(x) \!\).

The statement \( g'(x) = f(x) \) can be restated in the equivalent, and often more insightful, form

\[ f(x) = \frac{d}{dx} \int_a^x f(t) \, dt. \]

This is the most explicit way of stating the intuition of the main result, that differentiation reverses integration.

**Example 1:** Find the derivative \( g'(x) \) of the function

\[ g(x) = \int_0^x \frac{\sin(t)}{t} \, dt. \]

**Solution:** We have that

\[ g'(x) = \frac{d}{dx} \int_0^x \frac{\sin(t)}{t} \, dt = \frac{\sin(x)}{x}. \]
by the First Fundamental Theorem of Calculus.

**Example 2:** Find the derivative \( g'(x) \) of the function

\[
g(x) = \int_{-5}^{x} t^2 \ln(\pi t + 7) \, dt.
\]

**Solution:** When applying the First Fundamental Theorem of Calculus, it does not matter what the lower bound is. Consequently, we have that

\[
g'(x) = \frac{d}{dx} \int_{-5}^{x} t^2 \ln(\pi t + 7) \, dt = x^2 \ln(\pi x + 7).
\]

Now we ask the reverse question. If we differentiate a function, does integration recover the original function? And if so, in what manner does this manifest itself?

Consider the function \( F(x) \) given by

\[
F(x) = \int_{a}^{x} f(x) \, dx.
\]

We know from the First Fundamental Theorem of Calculus that \( F'(x) = f(x) \). In other words, \( F(x) \) is an antiderivative of \( f(x) \)! This means that antidifferentiation and integration (in the form of Riemann sums) were essentially the same process. Now consider what happens when we take the fixed bounds of integration \( a \) and \( b \). We have

\[
F(b) - F(a) = \int_{a}^{b} f(x) \, dx - \int_{a}^{a} f(x) \, dx = \int_{a}^{b} f(x) \, dx
\]

since the second integral has no width (and therefore zero area). This gives rise to the **Second Fundamental Theorem of Calculus**.

**Theorem 0.2** (Second Fundamental Theorem of Calculus). Suppose \( f \) is continuous on \([a, b]\). Then

\[
\int_{a}^{b} f(x) \, dx = F(b) - F(a)
\]

where \( F \) is the antiderivative of \( f \) (i.e. \( F'(x) = f(x) \)).

In other words, as we expected, integration reverses differentiation. More importantly, however, is that we now have a way to integrate which does not rely on Riemann sums. All we need to do is find the antiderivative of
the given function, and evaluate that at the upper and lower bounds \( a \) and \( b \! \)

**Example 3:** Reconsider the question of finding the area under the curve \( f(x) = x^2 \) between \( x = 0 \) and \( x = 1 \). Solve using the Second Fundamental Theorem of Calculus.

**Solution:** We can now state this as the integral question

\[
\int_0^1 x^2 \, dx
\]

and solve it using antidifferentiation. We know that the antiderivative of \( f(x) = x^2 \) is

\[
F(x) = \frac{x^3}{3}.
\]

We use the Second Fundamental Theorem of Calculus. The standard notation for the solution is

\[
\int_0^1 x^2 \, dx = \left[ \frac{x^3}{3} \right]_0^1 = \left( \frac{1}{3} \right) - \left( \frac{0}{3} \right) = \frac{1}{3}.
\]

In other words, we have very easily recovered the result from last week.

**Example 4:** Evaluate the definite integral

\[
\int_0^1 \sin(\pi t) \, dt.
\]

**Solution:** We have \( \frac{d}{dt} \cos(\pi t) = -\pi \sin(\pi t) \) so that

\[
\int_0^1 \sin(\pi t) \, dt = \left[ -\frac{\cos(\pi t)}{\pi} \right]_0^1 = \left[ -\frac{\cos(\pi)}{\pi} + \frac{\cos(0)}{\pi} \right] = \frac{2}{\pi}.
\]

**Example 5:** Evaluate

\[
\frac{d}{dx} \int_1^x \frac{1}{t} \, dt.
\]
Solution: We are tempted to use the First Fundamental Theorem of Calculus; however, the upper bound is $x^2$, not $x$, so this integral does not fit the required form for the theorem.

We can, however, solve this directly using the Second Fundamental Theorem. Since $F(t) = \ln(t)$ satisfies $F'(t) = \frac{1}{t}$, by the Second Fundamental Theorem of Calculus, we have

$$\frac{d}{dx} \int_{1}^{x^2} \frac{1}{t} \, dt = \frac{d}{dx} [\ln(t)]_{1}^{x^2}$$

$$= \frac{d}{dx} [\ln(x^2) - \ln(1)]$$

$$= \frac{d}{dx} \ln(x^2)$$

$$= \frac{2}{x}.$$

This answer is different than what we would have gotten if we had applied the First Fundamental Theorem of Calculus. Clearly, for integrals with bounds other than a constant to $x$, differentiation “undoes” integration in a different way than presented in the First Fundamental Theorem. It is not clear yet, however, what this difference is. To answer this, we consider the general form

$$\frac{d}{dx} \int_{g(x)}^{h(x)} f(t) \, dt. \quad (2)$$

That is to say, consider taking an integral where both the lower and upper bounds vary in $x$, and then taking the derivative with respect to $x$. Example 5 falls into this category, taking $g(x) = x^2$ and $h(x) = 1$.

Suppose that $F(x)$ is an antiderivative of $f(x)$, i.e. a function such that $F'(x) = f(x)$. We can use the Second Fundamental Theorem of Calculus to give

$$\frac{d}{dx} \int_{g(x)}^{h(x)} f(t) \, dt = \frac{d}{dx} [F(g(x)) - F(h(x))] \quad \text{(FTCII)}$$

$$= F'(g(x))g'(x) - F'(h(x))h'(x) \quad \text{(Chain rule)}$$

$$= f(g(x))g'(x) - f(h(x))h'(x).$$

In other words, to evaluate a derivative of an integral of the form (2) we need to evaluate the function at the upper bound ($g(x)$) then take the derivative of that bound, and evaluate the function at the lower bound ($h(x)$) then take the derivative of that bound. This is called **Liebniz’s Rule.**
It is important to note that we do not need to know the antiderivative $F(x)$ in order to evaluate (2). That is to say, we can evaluate these integrals even when the function itself cannot be integrated!

**Example 6:** Evaluate

$$\frac{d}{dx} \int_x^{x^2} t \, dt.$$

**Solution:** We have that $h(x) = x$, $g(x) = x^2$, and $f(t) = t$. It follows that $h'(x) = 1$, $g'(x) = 2x$, $f(h(x)) = x$, and $f(g(x)) = x^2$, so that

$$\frac{d}{dx} \int_x^{x^2} t \, dt = x^2(2x) - x(1) = 2x^3 - x.$$

**Example 7:** Find $p'(x)$ given that

$$p(x) = \int_{\sin(x)}^{2 - \cos(x)} e^{-t^2} \, dt.$$

**Solution:** We have that $h(x) = \sin(x)$, $g(x) = 2 - \cos(x)$, and $f(x) = e^{-t^2}$. It follows that $h'(x) = \cos(x)$, $g'(x) = \sin(x)$, $f(h(x)) = e^{-\sin^2(x)}$, and $f(g(x)) = e^{-(2 - \cos(x))^2}$ so that

$$p(x) = \frac{d}{dx} \int_{\sin(x)}^{2 - \cos(x)} e^{-t^2} \, dt = e^{-(2 - \cos(x))^2} \sin(x) - e^{-\sin^2(x)} \cos(x).$$

It is interesting to note that we did not have to compute the antiderivative of $e^{-t^2}$ in order to evaluate this. This is particular fortunate for this example... because $e^{-t^2}$ does not have one!