Area between curves

The Second Fundamental Theorem of Calculus gives us a good handle on how to evaluate the area bound between a function and the $x$-axis. But why should we be interested in the $x$-axis specifically? Is that always a good reference point for the relevant areas we are trying to compute?

Of course not! In many applications, we will be interested in more general areas. One such case is when we are interested in the area between curves. For examples, consider the question of asking how much area is bound by the curves $y = -x^2 + 2$ and $y = x^2$ (see Figure 1). How do we approach such a question?

![Figure 1: Area bound between $y = -x^2 + 2$ and $y = x^2$.](image)

We can see that we are only interested in the interval of integration $x = -1$ and $x = 1$. Now consider the area bound between $-x^2 + 2$ and the $x$-axis from $x = -1$ to $x = 1$. Call this area $A_1$. Also, call by $A_2$ the area
bound between $x^2$ and the $x$-axis from $x = -1$ and $x = 1$. Clearly we have

$$A = A_1 - A_2.$$ 

It follows immediately that

$$[\text{Area bound by curves}] = A_1 - A_2$$

$$= \int_{-1}^{1} (-x^2 + 2) \, dx - \int_{-1}^{1} x^2 \, dx$$

$$= \int_{-1}^{1} [(−x^2 + 2) − (x^2)] \, dx.$$ 

In other words, the integral simplifies to the upper function minus the lower function!

In general, the formula for the area bound by two curves $f(x)$ and $g(x)$ from $x = a$ to $x = b$, where $f(x) \geq g(x)$ on $x \in [a, b]$, is given by

$$\text{Area} = \int_{a}^{b} [f(x) − g(x)] \, dx.$$ 

**Example 1:** Find the area bound by $y = -x^2 + 2$ and $y = x^2$.

**Solution:** We have

$$\int_{a}^{b} [f(x) − g(x)] \, dx = \int_{-1}^{1} [(−x^2 + 2) − (x^2)] \, dx$$

$$= 2 \int_{-1}^{1} [1 - x^2] \, dx$$

$$= 2 \left[ x - \frac{x^3}{3} \right]_{-1}^{1}$$

$$= 2 \left[ \left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right) \right]$$

$$= 2 \left[ \frac{4}{3} \right] = \frac{8}{3}.$$ 

So the area bound between the curves is $8/3$!

Sometimes alternative considerations need to be made. Consider the following example.
Example 2: Find the area bound by $x - y^2 + 1 = 0$ and $x + y - 1 = 0$.

Solution: We need to find the end points of the relevant areas. Is it clear that the left endpoint is going to be $x = -1$. The right endpoint is given by one of the intersection points of the first curve and the second. We solve for $y$ in the second equation to get $y = -x + 1$. We substitute this into the first equation to get

$$x - y^2 + 1 = x - (-x+1)^2 + 1 = x - (x^2 - 2x+1)+1 = -x^2 + 3x = x(3-x) = 0$$

so that the intersections occur at $x = 0$ and $x = 3$. The corresponding $y$ values can be found from either curve. We have $x = 0$ implies $y = 1$ and $x = 3$ implies $y = -2$. The right-most point is $x = 3$, so that is our upper bound of integration.

We notice, however, that the upper function changes depending on where we are in the interval $[-1, 3]$. In the interval $[-1, 0]$ we the upper function is the upper half of $x - y^2 + 1 = 0$ and the lower function is the lower half of $x - y^2 + 1 = 0$. In the interval $[0, 3]$ the upper function is $x + y - 1 = 0$ and the lower function is the lower half of $x - y^2 + 1 = 0$ (see Figure 2).
We can compute the area by taking the sum of $A_1$ and $A_2$ to get

\[
\text{[Area]} = A_1 + A_2 \\
= \int_{-1}^{0} [\sqrt{x-1} - (-\sqrt{x+1})] \, dx + \int_{0}^{3} [(1-x) - (-\sqrt{x+1})] \, dx \\
= 2 \int_{-1}^{0} \sqrt{x+1} \, dx + \int_{0}^{3} \left[ \sqrt{x+1} - x + 1 \right] \, dx \\
= 2 \left[ \frac{2}{3} (x+1)^{3/2} \right]_{-1}^{0} + \left[ \frac{2}{3} (x+1)^{3/2} - \frac{x^2}{2} + x \right]_{0}^{3} \\
= \frac{4}{3} + \left[ \frac{2}{3} (4)^{3/2} - \frac{3^2}{2} + 3 - \frac{2}{3} \right] = \frac{9}{2}.
\]

This was a little bit of work, but we have managed to get to the answer. Consider, however, an alternative approach to the problem. Rather than imagining the area being given by thin rectangles stacked in the $x$ direction and extending in the $y$ direction, imagine them as thin rectangles stacked in the $y$ direction and extending the $x$ direction (see Figure 3). If we take the limit as the width of these rectangles goes to zero, this amounts to integrating the area over the $y$ direction instead of the $x$ direction. We notice that, in this direction, there is only one area to compute!

Figure 3: We can approximate the area bound between $x - y^2 + 1 = 0$ and $x + y - 1 = 0$ by rectangles running in either the $x$ or $y$ direction.

In order to do this, we need to solve for the upper $x$ values and the lower
$x$ values in terms of $y$ and integrate over $y$. We have

\[
\text{[Area]} = \int_{-2}^{1} \left[ (1 - y) - (-1 + y^2) \right] \, dy
\]

\[
= \int_{-2}^{1} \left[ 2 - y - y^2 \right] \, dy
\]

\[
= \left[ 2y - \frac{y^2}{2} - \frac{y^3}{3} \right]_{-2}^{1}
\]

\[
= \left[ 2 - \frac{1}{2} - \frac{1}{3} \right] - \left[ -4 - 2 + \frac{8}{3} \right]
\]

\[
= \frac{9}{2}
\]

In other words, we can integrate in either $x$ or $y$ direction. In either case, the process is the same: we find the upper and lower function and integrate over the remaining variable.