1 Completing the Square

So far we have learned how to handle integrals with trouble terms of the form $a^2 - b^2 x^2$, $a^2 + b^2 x^2$, and $b^2 x^2 - a^2$ by using trigonometric substitutions. The common thread with these terms was that there was a constant term ($a^2$) and a second-order term ($b^2 x^2$) (i.e. $x$ to the second power). Which substitution we used depended on which of these terms had a positive or negative sign.

We might ask whether we can also handle terms with a first-order term, i.e. things like $ax^2 + bx + c$. It turns out the apparent problem is just a matter of representation. If we were asked, for instance, to graph the parabola $y = ax^2 + bx + c$, we would immediately rearrange the expression into the form $y = m(x - p)^2 + q$ with new constants $m$, $p$ and $q$. From this form, we could immediately determine the vertex of the parabola and would know whether the parabola opened up or down.

The important thing to notice for our purposes is that

$$m(x - p)^2 + q$$

looks an awful lot like the troublesome terms we were solving for earlier this week in that there is one constant term $q$ and one second-order term $m(x - p)^2$. In fact, we can solve integrals containing troublesome terms with a first-order term in exactly this manner. First we represent the term in its factored form with just a constant and second-order term, and then we make the exactly same trigonometric substitution as we have been (e.g. $(x - p) = \sin(u)$, $(x - p) = \tan(u)$, etc.). The process by which we factor such terms is *completing the square.*

To complete the square for the term $ax^2 + bx + c$, we need to:

1. factor out the leading-order term $a$ to get $a \left( x^2 + \frac{b}{a} x \right) + c$

2. take half of the first-order term, square it, and add it and subtract it from the expression, to get $a \left( x^2 + \frac{b}{a} x + \frac{b^2}{4a^2} \right) + c - \frac{b^2}{4a}$
3. factor the square to get \( a \left( x + \frac{b}{2a} \right)^2 + \left( c - \frac{b^2}{4a} \right) \).

**Example 1:**

Evaluate \( \int \frac{x}{\sqrt{27 + 6x - x^2}} \, dx \).

We first complete the square. We have that

\[
27 + 6x - x^2 = 27 + 9 - (x^2 - 6x + 9) = 36 - (x - 3)^2.
\]

Since the second-order term is negative, we recognize this as a \( \sin(u) \) substitution. In order to be able to factor a complete \( 1 - \cos^2(u) \) term from the expression, we need to adjust the substitution by a constant. We use \( (x-3) = 6 \sin(u) \) (or, alternatively, \( x = 6 \sin(u) + 3 \) or \( u = \arcsin((x-3)/6) \)).

This gives

\[
\int \frac{x}{\sqrt{27 + 6x - x^2}} \, dx = \int \frac{x}{\sqrt{36 - (x-3)^2}} \, dx = \int \frac{6 \sin(u) + 3}{\sqrt{36 - 36 \sin^2(u)}} [6 \cos(u) \, du] = \int (6 \sin(u) + 3) \, du = -6 \cos(u) + 3u + C = -6 \cos \left( \arcsin \left( \frac{x - 3}{6} \right) \right) + 3 \arcsin \left( \frac{x - 3}{6} \right) + C = -\sqrt{36 - (x - 3)^2} + 3 \arcsin \left( \frac{x - 3}{6} \right) + C.
\]

**Alternative Approach**

We notice that if we were to try \( u = 36 - (x - 3)^2 \) as a substitution, we would have \( du = -2(x - 3) \, dx \). This implies another approach to the above integral, namely, splitting the integral by separating out a factor of
\[-2(x - 3). \text{ We will see whether this leads to an easier form.}\]

\[
\int \frac{x}{\sqrt{36 - (x - 3)^2}} \, dx = -\frac{1}{2} \int \frac{-2(x - 3)}{\sqrt{36 - (x - 3)^2}} \, dx + \int \frac{3}{\sqrt{36 - (x - 3)^2}} \, dx.
\]

We can evaluate the two integrals separately. For the first integral, we set \(u = 36 - (x - 3)^2\) so that \(du = -2(x - 3) \, dx\). We have

\[
-\frac{1}{2} \int \frac{-2(x - 3)}{\sqrt{36 - (x - 3)^2}} \, dx = -\frac{1}{2} \int \frac{1}{u^{1/2}} \, du = -u^{1/2} + C_1 = -\sqrt{36 - (x - 3)^2} + C_1.
\]

For the second integral, we substitute \(x - 3 = 6 \sin(u)\) (or \(u = \arcsin((x - 3)/6)\)) so that \(dx = 6 \cos(u) \, du\). We have

\[
3 \int \frac{1}{\sqrt{36 - (x - 3)^2}} \, dx = 3 \int \frac{1}{\sqrt{36 - 36 \sin^2(u)}} (6 \cos(u) \, du) = 3 \int \frac{6 \cos(u)}{6 \cos u} \, du = 3 \int 1 \, du = 3u + C_2 = 3 \arcsin \left( \frac{x - 3}{6} \right) + C_2.
\]

Putting everything together, we have

\[
\int \frac{x}{\sqrt{36 - (x - 3)^2}} \, dx = -\sqrt{36 - (x - 3)^2} + 3 \arcsin \left( \frac{x - 3}{6} \right) + C
\]
as before. Recognizing that part of an integral can be evaluated using something other than a trigonometric substitution is often helpful, although in this case, both methods are equally valid and (seem to) take about the same amount of work.

**Example 2:**

Evaluate

\[
\int \frac{\sqrt{x^2 + 2x - 3}}{x + 1} \, dx
\]
We complete the square to get
\[ x^2 + 2x - 3 = x^2 + 2x + 1 - 1 - 3 = (x + 1)^2 - 4. \]

Since the constant term is negative, we recognize this as a \( \sec(u) \) substitution. Again we need to account for constants, so we use \( x + 1 = 2 \sec(u) \) (or \( u = \arcsec((x + 1)/2) \)). We have
\[
\int \frac{\sqrt{x^2 + 2x - 3}}{x + 1} \, dx
= \frac{\sqrt{4 \sec^2(u) - 4}}{2 \sec(u)} \left[ 2 \sec(u) \tan(u) \, du \right]
= 2 \int \tan^2(u) \, du
= 2 \int [\sec^2(u) - 1] \, du
= 2 \tan(u) - 2u + C
= 2 \tan \left( \arcsec \left( \frac{x + 1}{2} \right) \right) - 2 \arcsec \left( \frac{x + 1}{2} \right) + C
= \sqrt{(x + 1)^2 - 4} - 2 \arcsec \left( \frac{x + 1}{2} \right) + C.
\]