1 Rational Integrals

Consider being asked to solve
\[ \int \frac{x}{(x-1)(x+1)(x+3)} \, dx. \]

We may be tempted to try a variety of different substitutions; however, there is a significantly simpler method by which this integral (and those like it) can be solved. The method is called partial fraction decomposition.

What we notice here is that if any one of the three components on the bottom appeared by itself, we would be able to integrate immediately. For instance, we can evaluate
\[ \int \frac{1}{x-1} \, dx = \ln |x-1| + C. \]

That was awfully easy. It would make our lives significantly easier if we could separate the term \(1/(x-1)(x+1)(x+3)\) into three fractions of the form, say, \(A/(x-1)\), \(B/(x+1)\), and \(C/(x+3)\). There is nothing telling us that we are not allowed to do this, so let’s try it. We set
\[ \frac{x}{(x-1)(x+1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}. \]  

So how do we solve for the constants \(A\), \(B\) and \(C\)? We will go over the general methods for doing this in a moment, but for our purposes here, let’s just notice that we can multiply across by the denominator on the left-hand side to get
\[ x = A(x+1)(x+3) + B(x-1)(x+3) + C(x-1)(x+1). \]

We notice that, in order for (2) to be satisfied, it must be satisfied for all \(x\). This means that we can select any value of \(x\) we want to solve for the constants \(A\), \(B\) and \(C\)!
Which values of $x$ should we choose. We notice the brackets on the right-hand side have zeroes at the values $x = 1$, $x = -1$, and $x = -3$. It will simplify our algebra to use these values.

Setting $x = 1$ we obtain $1 = A(2)(6)$ which implies $A = \frac{1}{8}$.

Setting $x = -1$ we obtain $-1 = B(-2)(2)$ which implies $B = \frac{1}{4}$.

Setting $x = -3$ we obtain $3 = C(-4)(-2)$ which implies $C = -\frac{3}{8}$.

That was not nearly as painful as it could have been, but let’s keep in mind what we have actually done. This means that

$$\frac{x}{(x-1)(x+1)(x+3)} = \frac{1}{8(x-1)} + \frac{1}{4(x+1)} - \frac{3}{8(x+3)}$$

(Notice that we can check to see if this answer is valid—i.e. whether the process is valid—by simply finding a common denominator. The process is tedious, but we might as well go through it once to convince ourselves that our method works:

\[
\begin{align*}
\frac{1}{8(x-1)} + \frac{1}{4(x+1)} - \frac{3}{8(x+3)} &= \frac{1}{4} \left[ \frac{1}{2(x-1)} + \frac{1}{x+1} - \frac{3}{2(x+3)} \right] \\
&= \frac{1}{4} \left[ \frac{(x+1)(x+3) + 2(x-1)(x+3) - 3(x+1)(x-1)}{2(x-1)(x+1)(x+3)} \right] \\
&= \frac{1}{8} \left[ \frac{x^2 + 4x + 3 + 2x^2 + 4x - 6 - 3x^2 + 3}{(x-1)(x+1)(x+3)} \right] \\
&= \frac{x}{(x-1)(x+1)(x+3)}.
\end{align*}
\]

So our partial fraction separation checks out!)

The only remaining step is to integrate. If we have done things correctly, this should be the easiest part of the whole problem!

\[
\begin{align*}
\int \frac{x}{(x-1)(x+1)(x+3)} \, dx &= \int \frac{1}{8(x-1)} \, dx + \int \frac{1}{4(x+1)} \, dx - \int \frac{3}{8(x+3)} \, dx \\
&= \frac{1}{8} \ln |x-1| + \frac{1}{4} \ln |x+1| - \frac{3}{8} \ln |x+3| + C \\
&= \frac{1}{4} \ln \left| \frac{(x-1)^{1/2}(x+1)}{(x+3)^{3/2}} \right| + C.
\end{align*}
\]
This example suggests a general method by which we can simplify integrals of the form
\[ \int \frac{f(x)}{g(x)} \, dx \]

where \( f(x) \) and \( g(x) \) are polynomials, i.e. \( f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \) and \( g(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0 \). Naturally, the motivating example was a simplest possible case, but we are able to handle more complicated examples using the same principle in the following way:

1. We need \( g(x) \) to be higher order than \( f(x) \) (i.e. \( m > n \)). If this is not the case, we have to perform long division. We will assume from now on that \( g(x) \) is higher order than \( f(x) \).

2. Fully factor \( g(x) \). The Fundamental Theorem of Algebra (mathematicians love fundamental theorems!) guarantees that any polynomial can be factored uniquely into chains of terms of one of two forms:
\[
(ax + b)^n \quad \text{or} \quad (ax^2 + bx + c)^n.
\]

3. Perform partial fraction decomposition on \( \frac{f(x)}{g(x)} \). Since the denominator contains terms of one of only two possible forms, there are in fact only two cases we need to consider! Terms of the form \((ax + b)^n\) in \( g(x) \) lead to the terms
\[
\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \cdots + \frac{A_n}{(ax + b)^n} \quad (2)
\]
on the right-hand side. Terms of the form \((ax^2 + bx + c)^n\) lead to the terms
\[
\frac{B_1 x + C_1}{ax^2 + bx + c} + \frac{B_2 x + C_2}{(ax^2 + bx + c)^2} + \cdots + \frac{B_n x + C_n}{(ax^2 + bx + c)^n}. \quad (3)
\]

4. Integrate! We know how to integrate all of the terms in the above expressions with standard and trigonometric substitutions.

**Example 1:**

Set up the partial fraction expansion of
\[
\frac{1}{(x - 1)(2x + 3)^3(x^2 + x + 1)^2(3x^2 - x + 3)}.\]
Do not evaluate for the constants!

This is a straightforward application of (2) and (3). We have

\[
\frac{1}{(x - 1)(2x + 3)^3(x^2 + x + 1)^2(3x^2 - x + 3)} = \frac{A}{x - 1} + \frac{B}{2x + 3} + \frac{C}{(2x + 3)^2} + \frac{D}{(2x + 3)^3} + \frac{Ex + F}{x^2 + x + 1} + \frac{Gx + H}{(x^2 + x + 1)^2} + \frac{Ix + J}{3x^2 - x + 3}.
\]

Example 2:

Evaluate

\[
\int \frac{x^4 + x^3 + x^2 - x}{x^3 - 1} \, dx.
\]

We notice first of that the order of the numerator is higher than the denominator, so we need to perform long division. We can perform standard long division or synthetic division (for those who know it). We can also notice that we can factor

\[x^4 + x^3 + x^2 - x = x(x^3 - 1) + (x^3 - 1) + x^2 + 1 = (x + 1)(x^3 - 1) + x^2 + 1.\]

This implies immediately that

\[
\int \frac{x^4 + x^3 + x^2 - x}{x^3 - 1} \, dx = \int \frac{(x + 1)(x^3 - 1) + x^2 + 1}{x^3 - 1} \, dx = \int \left[ x + 1 + \frac{x^2 + 1}{x^3 - 1} \right] \, dx.
\]

We can readily integrate \(x + 1\), so we will focus our attention on the remaining fraction.

In order to perform a partial fraction decomposition we need to be able to factor \(x^3 - 1\) into a chain of terms of form (2) or (3). The decomposition is given by the difference of cubes formula

\[x^3 - 1 = (x - 1)(x^2 + x + 1)\]

where \(x^2 + x + 1\) cannot be further decomposed. We have

\[
\int \frac{x^2 + 1}{x^3 - 1} \, dx = \int \frac{x^2 + 1}{(x - 1)(x^2 + x + 1)} \, dx.
\]
We set-up our partial fraction decomposition for three variables $A$, $B$, and $C$ so that

\[
\frac{x^2 + 1}{(x - 1)(x^2 + x + 1)} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + x + 1}.
\]

Multiplying across by the denominator on the left-hand side we arrive at the more manageable form

\[
x^2 + 1 = A(x^2 + x + 1) + (Bx + C)(x - 1).
\]

We recall that in order for our partial fraction decomposition to be valid, the above expression must hold for all values of $x$. This suggests two alternative methods of solving for the constants.

1. Plug values of $x$ into the equation until you have enough expressions to solve for the variables. Particularly useful values of $x$ are those which eliminate brackets (e.g. if $(x - 2)$ appears factored several times, select $x = 2$).

2. Collect powers of $x$ on the right-hand side and then equate coefficients on the left-hand and right-hand side.

For illustrative purposes, we will perform both methods here.

To the first method, we select the values $x = 0$, $x = 1$ and $x = -1$.

Plugging $x = 0$ into the expression, we have $1 = A - C$ which implies $C = A - 1$. Plugging in $x = 1$ gives $2 = 3A$ which implies $A = \frac{2}{3}$, and therefore that $C = -\frac{1}{3}$. Plugging in $x = -1$ gives $2 = A - 2C + 2B$. We can plug in our known values of $A$ and $C$ and solve for $B$ to get $B = \frac{1}{3}$.

Alternative, we can expand our original expression to get

\[
x^2 + 1 = (A + B)x^2 + (A - B + C)x + (A - C).
\]

Equating the coefficients of the $x$ terms on the left-hand and right-hand side (realizing that $x^2 + 1 = (1)x^2 + (0)x + (1)$) gives the system of equations

\[
\begin{align*}
A + B &= 1 \\
A - B + C &= 0 \\
A - C &= 1.
\end{align*}
\]

For those who are familiar with matrix analysis, this can be solved through row reduction. Otherwise, we back substitute variables to get $C = A - 1$.
\[
A - B + C = 2A - B = 1 \Rightarrow B = 2A - 1 \Rightarrow A + B = 3A = 2 \Rightarrow A = \frac{2}{3} \Rightarrow B = \frac{1}{3} \Rightarrow C = -\frac{1}{3}.
\]

We put this together to get
\[
\int \frac{x^2 + 1}{(x - 1)(x^2 + x + 1)} \, dx = \int \frac{2}{3(x-1)} \, dx + \int \frac{x-1}{3(x^2 + x + 1)} \, dx.
\]

For the first integral, we have
\[
\int \frac{2}{3(x-1)} \, dx = \frac{2}{3} \ln |x - 1| + C_1.
\]

The second integral is more challenging. The substitution \( u = x^2 + x + 1 \) fails, so we recognize it as a form requiring us to complete the square and then solve by trigonometric substitution. In fact, we have two options: we can solve the integral directly using this trigonometric substitution, or we can split the integral into two parts, one solvable with the substitution \( u = x^2 + x + 1 \) and one solvable by a trigonometric substitution. Last week’s lecture notes give one example of how to do the latter procedure; however, I will not generally use this method. The alternative method has the advantage that we will not have to construct trigonometric triangles when we change back to our original variable \( x \).

We first complete the square:
\[
x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} = \frac{1}{4} ((2x + 1)^2 + 3).
\]

This implies the substitution \( 2x + 1 = \sqrt{3} \tan(u) \) (or \( u = \arctan((2x + 1)/\sqrt{3}) \)) so that \( dx = (\sqrt{3}/2) \sec^2(u) \, du \).
This implies that

\[ \frac{1}{3} \int \frac{x - 1}{x^2 + x + 1} \, dx = \frac{4}{3} \int \frac{x - 1}{(2x + 1)^2 + 3} \, dx = \frac{4}{3} \int \frac{\sqrt{3} \tan(u) - \frac{3}{2}}{3(1 + \tan^2(u))} \left[ \frac{\sqrt{3}}{2} \sec^2(u) \, du \right] = \frac{1}{3} \int \tan(u) \, du - \frac{1}{\sqrt{3}} \int 1 \, du = \frac{1}{3} \ln |\sec(u)| - \frac{1}{\sqrt{3}}u + C_2 = \frac{1}{3} \ln |\sec \left( \arctan \left( \frac{2x + 1}{\sqrt{3}} \right) \right)| - \frac{1}{\sqrt{3}} \arctan \left( \frac{2x + 1}{\sqrt{3}} \right) + C_2 = \frac{1}{3} \ln \left| \frac{(2x + 1)^2 + 3}{\sqrt{3}} \right| - \frac{1}{\sqrt{3}} \arctan \left( \frac{2x + 1}{\sqrt{3}} \right) + C_2 = \frac{1}{6} \ln |3 + (2x + 1)^2| - \frac{1}{\sqrt{3}} \arctan \left( \frac{2x + 1}{\sqrt{3}} \right) + C_2. \]

We can now put everything together to get

\[ \int \frac{x^4 + x^3 + x^2 - x}{x^3 + 1} \, dx = \frac{x^2}{2} + x + \frac{2}{3} \ln |x - 1| + \frac{1}{6} \ln |3 + (2x + 1)^2| - \frac{1}{\sqrt{3}} \arctan \left( \frac{2x + 1}{\sqrt{3}} \right) + C. \]

**Example 3:**

Evaluate

\[ \int \frac{-2x^2 - 4x + 1}{4x^4 - 4x^2 + 1} \, dx. \]

We notice that the denominator is higher-order than the numerator, so we do not need to worry about long-division. Our first step is to factor the denominator. We notice by difference of squares that

\[ 4x^4 - 4x^2 + 1 = (2x^2 - 1)^2. \]
We could factor this further by taking \((2x^2 - 1) = (\sqrt{2}x + 1)(\sqrt{2}x - 1)\); however, this will lead to more work than doing our partial fraction decomposition on this term directly. You can try the extra expansion if you like.

We consider the partial fraction decomposition

\[
\frac{-2x^2 - 4x + 1}{(2x^2 - 1)^2} = \frac{Ax - B}{2x^2 - 1} + \frac{Cx - D}{(2x^2 - 1)^2}.
\]

We rearrange this so that

\[
-2x^2 - 4x + 1 = (Ax - B)(2x^2 - 1) + Cx + D = Ax^3 + 2Bx^2 + [C - A]x + [D - B].
\]

We equate coefficients on the left- and right-hand sides to get \(A = 0\), \(B = -1\), \(C = -4\), and \(D = 0\). This leads to

\[
\int \frac{-2x^2 - 4x + 1}{(2x^2 - 1)^2} = \int \frac{1}{1 - 2x^2} \, dx - \int \frac{4x}{(2x^2 - 1)^2} \, dx.
\]

The first integral is clearly the form of a \(\sin(u)\) substitution. Adjusting for constants, we try \(x = (1/\sqrt{2}) \sin(u)\) (or \(u = \arcsin(\sqrt{2}x)\)) so that \(dx = (1/\sqrt{2}) \cos(u) \, du\). We have

\[
\int \frac{1}{1 - 2x^2} \, dx = \frac{1}{\sqrt{2}} \int \frac{1}{1 - \sin^2(u)} \cos(u) \, du
\]

\[
= \frac{1}{\sqrt{2}} \int \frac{1}{\cos(u)} \, du
\]

\[
= \frac{1}{\sqrt{2}} \int \sec(u) \, du
\]

\[
= \frac{1}{\sqrt{2}} \ln |\sec(u) + \tan(u)| + C_1
\]

\[
= \frac{1}{\sqrt{2}} \ln \left| \sec(\arcsin(\sqrt{2}x)) + \tan(\arcsin(\sqrt{2}x)) \right| + C_1
\]

\[
= \frac{1}{\sqrt{2}} \ln \left| \frac{1}{\sqrt{1-2x^2}} + \frac{\sqrt{2}x}{\sqrt{1-2x^2}} \right| + C_1
\]

\[
= \frac{1}{\sqrt{2}} \ln \left| \frac{1 - \sqrt{2}x}{\sqrt{1-2x^2}} \right| + C_1.
\]
We may be tempted to try a trigonometric substitution on the second integral. There is something else to notice here. We recall that trigonometric substitution is only necessary if a direct substitution on the troublesome term fails. In this case, if we set $u = 2x^2 - 1$ we have $du = 4x \, dx$ which eliminates the numerator! We have
\[
\int \frac{4x}{(2x^2 - 1)^2} \, dx = \int \frac{1}{u^2} \, du = -\frac{1}{u} + C_2 = -\frac{1}{2x^2 - 1} + C_2.
\]

Putting everything together, we have
\[
\int \frac{-2x^2 - 4x + 1}{4x^4 - 4x^2 + 1} \, dx = \frac{1}{\sqrt{2}} \ln \left| \frac{1 - \sqrt{2}x}{\sqrt{1 - 2x^2}} \right| + \frac{1}{2x^2 - 1} + C.
\]

**Example 4:**

Reconsider the following example from last week’s notes:
\[
\int \frac{1}{x^2 - 1} \, dx.
\]

We solved this previously using a sec($u$) substitution, but it turns out there is another (and easier!) way to solve it. We notice that $x^2 - 1$ is a difference of squares, which can be factored as
\[
x^2 - 1 = (x + 1)(x - 1).
\]

This is exactly the form of denominator we are able to separate using partial fractions.

We set
\[
\frac{1}{x^2 - 1} = \frac{1}{(x + 1)(x - 1)} = \frac{A}{x + 1} + \frac{B}{x - 1}
\]
to get
\[
1 = A(x - 1) + B(x + 1).
\]

Setting $x = -1$ we obtain $1 = -2A$ which implies $A = -\frac{1}{2}$.

Setting $x = 1$ we obtain $1 = 2B$ which implies $B = \frac{1}{2}$.
Our integral can now be solved:

\[
\int \frac{1}{x^2 - 1} \, dx = -\frac{1}{2} \int \frac{1}{x + 1} \, dx + \frac{1}{2} \int \frac{1}{x - 1} \, dx \\
= -\frac{1}{2} \ln |x + 1| + \frac{1}{2} \ln |x - 1| + C \\
= \frac{1}{2} \ln \left| \frac{x - 1}{x + 1} \right| + C.
\]

This is certainly a sufficient and correct answer, so normally we would stop here; however, in order to see that it is, in fact, the same as answer as we obtained when we solved using a \( \sec(u) \) substitution, we perform a little additional manipulation:

\[
\frac{1}{2} \ln \left| \frac{x - 1}{x + 1} \right| + C \\
= \ln \left| \sqrt{\frac{x - 1}{x + 1}} \right| + C \\
= \ln \left| \sqrt{\frac{x - 1}{x + 1}} \sqrt{\frac{x + 1}{x + 1}} \right| + C \\
= \ln \left| \sqrt{x^2 - 1} \right| + C.
\]

This is exactly the form we obtained last week. It is not uncommon that integration problems can be solve in multiple ways. In general, there is no preference for which method to use (unless the question says to use a specific method), but some will often be easier than others.