SYDE, LECTURE 15:
Arc Length

1 Arc Length

Suppose a wire is bent according to the function $f(x) = x^{3/2}$ for $0 \leq x \leq 1$ (see Figure 1(a)). We can see that the distance between the endpoints of the wire is one unit in the $x$ direction and one unit in the $y$ direction, for a total distance of $\sqrt{2}$ units, but how long is the actual wire?

![Figure 1:](image)

This is a question of determining the **arc length** of a curve. It should come as no surprise that the answer to the question involves methods outlined so far in this course, namely, differentiation and integration.

Consider dividing the interval $a \leq x \leq b$ into $n$ subintervals with equal width $\Delta x$ (see Figure 1(b)). For a small $\Delta x$, we can approximate the length of the curve between $f(x_i)$ and $f(x_{i+1})$ by the length of the straight line between the two points (see Figure 1(c)). This can be solved for by Pythagoras’ Theorem. For the $i^{th}$ line segment, we have sides lengths $\Delta x$ and $f(x_{i+1}) - f(x_i)$ so that the length of the line segment connecting $f(x_i)$ and $f(x_{i+1})$ is

$$L_i = \sqrt{(\Delta x)^2 + (f(x_{i+1}) - f(x_i))^2}.$$

Since we imagine the length of the curve can be well-approximated in this
way, we have

\[ \text{Arc length} = \sum_{i=1}^{n} L_i \]

\[ = \sum_{i=1}^{n} \sqrt{(\Delta x)^2 + (f(x_{i+1}) - f(x_i))^2} \]

\[ = \sum_{i=1}^{n} \sqrt{1 + \left( \frac{f(x_{i+1}) - f(x_i)}{\Delta x} \right)^2} \Delta x. \]

This is just an approximation, but we expect that the approximation will get “better” and “better” as we take “smaller” and “smaller” intervals. Two interesting things happen when we take the limit as \( n \to \infty \). We have, first of all, that

\[ \frac{f(x_{i+1}) - f(x_i)}{\Delta x} \to f'(x). \]

We also have that

\[ \sum_{i=1}^{n} [\text{stuff}] \Delta x \to \int_{a}^{b} [\text{stuff}] \, dx. \]

It follows that the arc length of a curve evaluated from \( x = a \) to \( x = b \) is given by

\[ \text{Arc length} = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{1 + \left( \frac{f(x_{i+1}) - f(x_i)}{\Delta x} \right)^2} \Delta x \]

\[ = \int_{a}^{b} \sqrt{1 + (f'(x))^2} \, dx. \]

For our problem, we have \( f(x) = x^{3/2} \), which implies \( f'(x) = 3/2 x^{1/2} \). It follows that we have

\[ \text{Arc length} = \int_{0}^{1} \sqrt{1 + \left( \frac{3}{2} x^{1/2} \right)^2} \, dx \]

\[ = \int_{0}^{1} \sqrt{1 + \frac{9}{4} x} \, dx \]

\[ = \frac{8}{27} \left( \frac{1}{4} + \frac{9}{4} x \right)^{3/2} \Bigg|_{0}^{1} = \frac{8}{27} \left[ \left( \frac{13}{4} \right)^{3/2} - 1 \right] \approx 1.4397. \]
It is worth noting that, in general, it is very hard to compute arc lengths exactly. Examples which work out exactly need to be carefully manufactured to work out as they do (and they will for the purposes of this class!). In practice, arc lengths are estimated (very accurately!) using computers.

**Example 1:** Determine the arc length of \( f(x) = \ln(\sec(x)) \) between \( 0 \leq x \leq \pi/4 \).

**Solution:** We have
\[
f'(x) = \frac{1}{\sec(x)} \cdot \sec(x) \tan(x) = \tan(x)
\]
so that \([f'(x)]^2 = \tan^2(x)\). It follows from the arc length formula that we have
\[
\text{Arc length} = \int_0^{\pi/4} \sqrt{1 + \tan^2(x)} \, dx
\]
\[
= \int_0^{\pi/4} \sqrt{\sec^2(x)} \, dx
\]
\[
= \int_0^{\pi/4} \sec(x) \, dx
\]
\[
= [\ln(\sec(x) + \tan(x))]_0^{\pi/4}
\]
\[
= \ln(\sqrt{2} + 1) 
\approx 0.8814.
\]

**Example 2:** Determine the arc length of \( f(x) = (1/6)x^3 + (1/2)x^{-1} \) between \( 1 \leq x \leq 2 \).

**Solution:** We have \( f'(x) = (1/2)x^2 - (1/2)x^{-2} \). It follows by the formula
that we have

\[
\text{Arc length} = \int_1^2 \sqrt{1 + \left(\frac{1}{2}x^2 - \frac{1}{2}x^{-2}\right)^2} \, dx \\
= \int_1^2 \sqrt{1 + \frac{1}{4}x^4 - \frac{1}{2} + \frac{1}{4}x^{-4}} \, dx \\
= \int_1^2 \sqrt{\frac{1}{4} + \frac{1}{2} + \frac{1}{4}x^{-4}} \, dx \\
= \int_1^2 \sqrt{\left(\frac{1}{2}x^2 + \frac{1}{2}x^{-2}\right)^2} \, dx \\
= \int_1^2 \left(\frac{1}{2}x^2 + \frac{1}{2}x^{-2}\right) \, dx \\
= \left[\frac{1}{6}x^3 - \frac{1}{2}x^{-1}\right]_1 \\
= \left(\frac{8}{6} - \frac{1}{4}\right) - \left(\frac{1}{6} - \frac{1}{2}\right) = \frac{17}{12}.
\]