1 Mass and density

We know that the mass or something corresponds to how much of that something there is, which the density corresponds to how much of that something there is per unit volume. Examples of contrasts between mass and density include:

1. How many people live in a certain area (the population) versus where the people are clustered (the population density) (e.g. rural versus urban areas).

2. How much a beam weighs (its mass) versus where the weight is distributed throughout its length (its density).

3. The overall amount of a solvent or solute in a mixture versus its spatial distribution (relative concentration).

If a solid body has a constant density profile $\rho(x) = \rho$, then the mass is just given by density times the volume. That is to say, $M = \rho V$. If the density profile varies over the domain, we no longer have this intuition. However, if we break the domain into small pieces (with volume $V_i$), we can estimate the mass in that area by taking $M_i \approx \rho(x_i)V_i$. It follows that, if the domain is divided into $n$ regions, we can estimate the overall mass by taking

$$M \approx \sum_{i=1}^{n} \rho(x_i)V_i.$$

Taking the limit as $n$ goes to infinity, we have

$$M = \int_{D} \rho(x) \, dV.$$

There are a few things worth noting about this formula:

1. In general, the domain $D$ can be over multiple variables. We, however, have not dealt with this case (yet!) so we will consider density profiles which can be reduced to a single variable.
2. When $\rho(x) = \rho$ is just a constant, this reduces to our earlier formula.

**Example 1:** Consider a circular pipe of fixed radius 1m with the concentration density profile for a particular toxin given by $\rho(x) = (-x^2 + 10x) \text{kg/m}^3$, $0 \leq x \leq 10$, where $x$ is the distance down the pipe (in m). How much toxin is in the pipe?

**Solution:** The amount of toxin in a small length $\Delta x$ of the pipe can be approximated by the density at some point in the slice, averaged over the cross-sectional area times the length of the pipe. If we sum over all such segments, we have that the mass can be approximated by

$$M \approx \sum_{i=1}^{n} f(x_i)\pi \Delta x.$$

Following the intuition we have been relying on so much for this course, we image that taking the limit as $\Delta x$ goes to zero will given the actual mass, so that we have

$$M = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)(\pi) \Delta x$$

$$= \pi \int_{0}^{10} (-x^2 + 10x) \, dx$$

$$= \pi \left( -\frac{x^3}{3} + 5x^2 \right)_0^{10}$$

$$= \pi \left( -\frac{1000}{3} + 500 \right)$$

$$= \frac{500\pi}{3} \text{ kg.}$$

### 2 Centre of Mass

Consider the question of where we should balance a wire with variable density in order that it does not tip over to one side or the other. This is a question of finding an object’s **centre of mass**.

In order to find an objects centre of mass, we need to find a point $x^* \in [a, b]$ such that

$$\int_{a}^{b} (x - x^*)\rho(x) \, dx = 0.$$
That is to say, we need to find a point such that the mass “to the left” and
the mass “to the right” balance out. We can rearrange this equation to give
\[ x^* = \frac{\int_a^b x \rho(x) \, dx}{\int_a^b \rho(x) \, dx}. \]

**Example 1:** Consider a wire with density profile \( \rho(x) = \sin(x) \), \( 0 \leq x \leq \pi/2 \). Find the centre of mass.

**Solution:** We have
\[
\int_0^{\pi/2} \sin(x) \, dx = [-\cos(x)]_0^{\pi/2} = -\cos\left(\frac{\pi}{2}\right) + \cos(0) = 1.
\]
We also have
\[
\int_0^{\pi/2} x \sin(x) \, dx = [-x \cos(x) + \sin(x)]_0^{\pi/2} = -\frac{\pi}{2} \cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right) = 1.
\]

It follows that the center of mass is given by
\[ x^* = \frac{\int_a^b x \rho(x) \, dx}{\int_a^b \rho(x) \, dx} = 1. \]

### 3 Centroids

If we consider the density of a shape to be uniform, the centre of mass corresponds to the geometrical centre—that is to say, it depends on the shape of the region alone. This is called the centroid of the region.

**Theorem 3.1.** Consider the region bound by \( a \leq x \leq b \) and \( 0 \leq y \leq f(x) \). Then the centroid \((x^*, y^*)\) of the region is given by

\[
x^* = \frac{\int_a^b x f(x) \, dx}{A} \quad \text{and} \quad y^* = \frac{\frac{1}{2} \int_a^b f(x)^2 \, dx}{A}
\]
Figure 1: The centre of mass for a wire of length $\pi/4$ and density profile \( \rho(x) = \sin(x) \) is \( x^* = 1 \). That is to say, the density to the left of \( x^* = 1 \) and the density to the right of \( x^* = 1 \) are equal.

where

\[
A = \int_a^b f(x) \, dx.
\]

We make a few notes:

1. The quantity \( A \) is familiar—it is just the area under the curve \( f(x) \) between \( x = a \) and \( x = b \)!

2. We can easily generalize to areas not bound by the axis by considering \( f(x) - g(x) \) for \( f(x) \geq g(x) \). We have

\[
x^* = \frac{\int_a^b x(f(x) - g(x)) \, dx}{A} \quad \text{and} \quad y^* = \frac{\frac{1}{2} \int_a^b (f(x)^2 - g(x)^2) \, dx}{A}.
\]

**Example 1:** Find the centroid of the region bound by \( f(x) = e^{-x} \) and the \( x \)-axis between \( x = 0 \) and \( x = 1 \).

**Solution:** We have

\[
A = \int_0^1 e^{-x} \, dx = [-e^{-x}]_0^1 = 1 - \frac{1}{e} = \frac{e - 1}{e}.
\]

We also have

\[
\int_0^1 xe^{-x} \, dx = [-xe^{-x} - e^{-x}]_0^1 = 1 - \frac{2}{e} = \frac{e - 2}{e}.
\]
and
\[ \frac{1}{2} \int_0^1 e^{-2x} \, dx = \frac{1}{2} \left[ -\frac{1}{2} e^{-2x} \right]_0^1 = \frac{1}{4} \left( 1 - \frac{1}{e^2} \right) = \frac{e^2 - 1}{4e^2}. \]

It follows that
\[ x^* = \frac{e - 2}{e - 1} = \frac{e - 2}{e - 1} \approx 0.418 \]

and
\[ y^* = \frac{e^2 - 1}{4e^2} = \frac{(e + 1)(e - 1)}{4e^2} \cdot \frac{e}{e - 1} = \frac{e + 1}{4e} \approx 0.342. \]

Figure 2: The region below the curve \( f(x) = e^{-x} \) bound between \( x = 0 \) and \( x = 1 \) has centroid \((x^*, y^*) = (0.418, 0.342)\).

**Example 2:** Find the centroid of the region bound by \( f(x) = e^{-x} \) and the \( x \)-axis between \( x = 0 \) and \( x = \infty \).

**Solution:** We use our regular centroid formulas with the additional
wrinkle that the integrals are improper integrals. We have

\[ A = \int_{0}^{\infty} e^{-x} \, dx \]

\[ = \lim_{a \to \infty} \int_{0}^{a} e^{-x} \, dx \]

\[ = \lim_{a \to \infty} \left[ -e^{-x} \right]_{0}^{a} \]

\[ = \lim_{a \to \infty} \left[ -e^{-a} + 1 \right] \]

\[ = 1 \]

and

\[ \int_{0}^{\infty} xe^{-x} \, dx = \lim_{a \to \infty} \int_{0}^{a} xe^{-x} \, dx \]

\[ = \lim_{a \to \infty} \left( \left[ -xe^{-x} \right]_{0}^{a} + \int_{0}^{a} e^{-x} \, dx \right) \]

\[ = \lim_{a \to \infty} \left[ -xe^{-x} - e^{-x} \right]_{0}^{a} \]

\[ = \lim_{a \to \infty} \left[ -ae^{-a} - e^{-a} + 1 \right] \]

\[ = 1 \]

and

\[ \frac{1}{2} \int_{0}^{\infty} e^{-2x} \, dx = \frac{1}{2} \lim_{a \to \infty} \int_{0}^{a} e^{-2x} \, dx \]

\[ = \frac{1}{2} \lim_{a \to \infty} \left[ \frac{-e^{-2x}}{2} \right]_{0}^{a} \]

\[ = \frac{1}{2} \lim_{a \to \infty} \left[ -e^{-2a} + 1 \right] \]

\[ = \frac{1}{4} \]

It follows that

\[ x^* = 1, \quad \text{and} \quad y^* = \frac{1}{4}. \]

4 Pappus’s Theorem

A very important application of centroids is that they allow us to (relatively easily) compute the volumes and surface areas of shapes that we previously had to compute using one of our volumes of revolution techniques. The result which justifies this is called Pappus’s Theorem.
Theorem 4.1. **Volume:** Consider the shape produced by rotating a planar region \( R \) with centroid \((x^*, y^*)\) about a line. Then the volume of the shape is given by the product of the area of \( R \) multiplied by the distance travelled by the centroid under rotation; and

**Surface area:** Consider the shape produced by rotating a planar curve \( C \) with centroid \((x^*, y^*)\) about a line. The surface area of the shape is given by the product of the length of the curve \( C \) multiplied by the distance travelled by the centroid under rotation.

**Example 1:** Find the volume and surface area of the shape produced by rotating the circle centred at \((a, 0)\) of radius \( b < a \) around the \( y \)-axis (torus).

**Solution:** The centroid of the circle is obviously the centre of the circle \((a, 0)\). Rotating about the \( y \)-axis gives an overall length of travel under rotation of \( 2\pi a \). Since the area of the circle is \( \pi b^2 \) and the length of the curve enclosing the circle is \( 2\pi b \), we have by Pappus’s Theorem that

\[
\text{Volume} = (\pi b^2)(2\pi a) = 2\pi^2 ab^2
\]

and

\[
\text{Surface area} = (2\pi b)(2\pi a) = 4\pi^2 ab.
\]