1 Functions of Multiple Variables

So far, we have been dealing with functions of the form \( y = f(x) \). This is a function of a single variable, since we need only specify the value of a single independent variable, \( x \), in order to determine the value of the dependent variable, \( y \).

Certainly, the real world is more complicated than dependences like this. Imagine, for instance, a cartographer tasked with building an elevation map of a particular region (see Figure 1).

![Figure 1: The elevation of a topographical map depends on two variables, a longitude and a latitude.](image)

We can readily see that the dependent variable (say, \( z \)) is dependent on two variables, an \( x \) and a \( y \) (which correspond to the longitude and the latitude). If we only specify a longitude or a latitude, it is not sufficient to fully determine the elevation.
Many simple real-world applications depend on functions of multiple variables. For instance, the volume of a cylinder is given by

\[ V = \pi r^2 h. \]

We need to specify two variables, the radius and the height, in order to determine the volume—specifying one or the other is not sufficient. Complexity can obviously grow from there. Consider the computing the profit of an international corporation, which depends sensitively on material costs, sales, wages, etc.—literally thousands of variables to depend on!

There are several things worth noting before proceeding:

1. Dependencies can be even more complicated than this. For instance, we could have multiple dependent variables. For the purposes of this course, however, we will typically be interested in multiple (usually two, for visualization purposes) independent variables but only one dependent variable.

2. In order for a dependent variable to be a function of more than one independent variable, a unique point must be given for each combination of the independent variables in the domain. This is the multidimensional analogue of the vertical line test.

3. The domain of a multivariate function \( f(x,y) \) (which we will still denote \( D(f) \)) now represents a region in space, rather than just an interval. All the normal restrictions to functions apply (e.g. cannot divide by zero, square roots must be nonnegative, etc.).

4. The range of a multivariable function \( f(x,y) \) (which we will still denote \( R(f) \)) is the set of values which may be obtained by inserting values in \( D(f) \) into \( f \).

5. It is common to give the output of a multivariate function a name. In the case where there are two independent variables and one dependent variable, it is most common to write \( z = f(x,y) \) and set up the axes so that the \( (x,y) \)-plane lies flat and the \( z \)-axis represents a height above this plane.

**Example 1:** Determine which of the following relationships can be stated in the functional form \( z = f(x,y) \):

(a) \( x + y - z = 0 \),  (b) \( xy - \ln(z) = 0 \),  (c) \( x^2 + y^2 + z^2 = 1 \).
Solution: For part (a), we can write $z = x + y$ so that $z = f(x, y)$ where $f(x, y) = x + y$. It follows that $z$ is a function of $x$ and $y$ (that is to say, we can uniquely determine a $z$ value given a specified $x$ and $y$ value). This function corresponds to a plane.

For part (b), we can rearrange to get

$$\ln(z) = xy, \quad \implies \quad z = e^{xy}$$

which satisfies the functional form $z = f(x, y)$ with $f(x, y) = e^{xy}$.

For part (c), we recognize the variable dependences as representing a sphere of radius 1. This is not a function because many pairs of points $(x, y)$ in the domain can be corresponded to multiple values $z$. For instance, if we solve

$$z^2 = 1 - x^2 - y^2, \quad \implies \quad z = \pm \sqrt{1 - x^2 - y^2}$$

we can immediately see that, for instance, the point $(x, y) = (0, 0)$ can be corresponded to the points $z = 1$ and $z = -1$.

Example 2: Find the domain and range of

$$f(x, y) = \sqrt{9 - x^2 - y^2}.$$ 

Solution: As in the single variable case, we need the value under the square root to be nonnegative, so that we require

$$9 - x^2 - y^2 \geq 0 \quad \implies \quad x^2 + y^2 \leq 9.$$ 

This is the circle of radius 3 centred at $(0, 0)$. If we pick any points outside of this disc, the multivariable function $f(x, y)$ is not defined (see Figure 2). The range is the set of values which can be obtained by entering values in this region into $f$. It is clear that $0 \leq x^2 + y^2 \leq 9$ so that $0 \leq 9 - x^2 - y^2 \leq 9$, which implies $0 \leq \sqrt{9 - x^2 - y^2} \leq 3$. It follows that the range of $f(x, y)$ is $[0, 3]$.

Example 3: Find the domain and range of

$$f(x, y) = \ln(2x - 5y).$$ 

Solution: The natural logarithm can only take in positive values, so that the domain is

$$2x - 5y > 0 \quad \implies \quad y < \frac{2}{5}x.$$ 

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Figure 2: The domain of \( f(x, y) = \sqrt{9 - x^2 - y^2} \) is the unit disc \( x^2 + y^2 \leq 9 \).

Since these values span over the entire domain of \( \ln(\cdot) \), we have that the range is \( \mathbb{R} \). (See figure 3.)

**Example 4:** Find the domain and range of \( f(x, y) = \frac{1}{x^2 + y^2} \).

**Solution:** The only exception we have here for the domain is that we cannot divide by zero. We can clearly see that \( x^2 + y^2 = 0 \) is satisfied if and only if \((x, y) = (0, 0)\) so that
\[
D(f) = \{ (x, y) \in \mathbb{R}^2 \mid (x, y) \neq (0, 0) \}.
\]

We notice that the function \( f(x, y) \) can never produce a negative value, or even the value zero (since this only occurs in the limit as \( x \to \infty \) or \( y \to \infty \)) so that we have
\[
R(f) = \{ z \in \mathbb{R} \mid z > 0 \}.
\]

## 2 Level Curves and Contour Plots

Consider the question of *inverting* a multivariable function. We approach this in the same as we did for single variable functions. Normally we ask, given a value in our domain (or point), what is the value of the function?
Figure 3: The domain of $f(x, y) = \ln(2x - 5y)$ is the half-plane $y < (2/5)x$. Note that this does not include the line $y = (2/5)x$.

Now we ask, given a value for the function, which values in the domain can attain it?

We can see very quickly that this is not as simple of a question as we might hope. Consider the earlier example

$$f(x, y) = \sqrt{9 - x^2 - y^2}.$$  

We know that the range of $f$ is the interval $[0, 3]$ so we might as well pick a few values from within here. We have

- $f(x, y) = 0 \implies 0 = \sqrt{9 - x^2 - y^2} \implies x^2 + y^2 = 9$
- $f(x, y) = 1 \implies 1 = \sqrt{9 - x^2 - y^2} \implies x^2 + y^2 = 8$
- $f(x, y) = 2 \implies 2 = \sqrt{9 - x^2 - y^2} \implies x^2 + y^2 = 5$
- $f(x, y) = 3 \implies 3 = \sqrt{9 - x^2 - y^2} \implies (x, y) = (0, 0)$.

It is clear that each value of the function does not correspond to a unique point in the domain of $f$—rather, in general it corresponds to a curve! If we pick any point along the curve $x^2 + y^2 = 2\sqrt{2}$, for instance, we obtain the value $f(x, y) = 1$. In general, the set of points satisfying the relation $f(x, y) = C$ for some constant $C$ is called a **level curve**.

We can plot this sample of level curves to get a **contour plot** (see Figure 4). There are a few notes worth making about contour plots:
1. Contour plots are one of the most important methods by which to visualize a multivariate function. They have the advantage of needing fewer dimensions than a full plot. Still, many properties of the original functions (e.g. continuity, steepness, max/min values) can be guessed based on the contour plots.

2. We have to be careful to select a representative sample of values $C$. If we choose too few points, or an unrepresentative sample of points, we may miss important qualitative behaviour. If we choose too many points, the plot may become cluttered.

3. Contour plots are common in everyday life! We use them on maps to illustrate elevation, precipitation, temperature, etc., over two-dimensional terrains.

![Figure 4: The plot of the level curves of the function $f(x, y) = \sqrt{9 - x^2 - y^2}$ is called the contour plot.](image)

**Example 1:** Draw a contour plot for the function

$$f(x, y) = x^2 - y^2.$$  

**Solution:** We set $f(x, y) = C$ to get

$$x^2 - y^2 = C.$$  

If $C = 0$, we obtain the lines $y = x$ and $y = -x$. If $C > 0$ we obtain hyperbolas going through $(\pm(1/\sqrt{C}), 0)$ and opening to the left and right.
If $C < 0$ we obtain hyperbolas going through $(0, \pm(1/\sqrt{C}))$ and opening up and down. (See Figure 5.)

![Contour plot](image)

Figure 5: Contour plot of the function $f(x, y) = x^2 - y^2$.

**Example 2:** Draw a contour plot for the function

$$f(x, y) = \frac{2xy}{x^2 + y^2}.$$

**Solution:** We set $f(x, y) = C$ to get

$$C = \frac{2xy}{x^2 + y^2} \implies y^2 - \frac{2}{C}xy + x^2 = 0.$$

We can solve this using the quadratic formula to get

$$y = \frac{2x \pm \sqrt{4x^2 - 4x^2}}{2} = \left(1 \pm \sqrt{1 - C^2}\right)x.$$

It follows that the level curves are straight lines through $(0, 0)$! We can also see that we need $1 - C^2 \geq 0$ which gives $-1 \leq C \leq 1$.

If we pick a representative sample of values for $C$ in the interval $-1 \leq C \leq 1$.
$C \leq 1$, we arrive at

\[
\begin{align*}
C = -1 & \implies y = -x \\
C = -1/2 & \implies y = (-2 \pm \sqrt{3})x \\
C = 0 & \implies x = 0 \text{ or } y = 0 \\
C = 1/2 & \implies y = (2 \pm \sqrt{3})x \\
C = 1 & \implies y = x.
\end{align*}
\]

It follows that the contour plot can be given by Figure 6.

Figure 6: Contour plot of the function $f(x, y) = \frac{2xy}{x^2 + y^2}$. 

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