Section 1: Review

This course will continue the development of the Calculus tools started in Math 30 and Math 31.

The primary difference between this course and previous ones is that, rather than considering traditional functions \( y = f(x) \) of one variable, we will consider functions of multiple variables. This extension should not come as a shock—most real-world applications cannot be reduced a relationship between a single input and a single output. The price of a stock depends on hundreds of market factors. The effectiveness of a drug depends on dozens of environmental and genetic controls. Even the space we live in fundamentally three-dimensional! If we wish to understand such systems, we will need to extend the tools we have developed so far to this multi-dimensional setting.

The rough outline for the course is as follows:

- Parametric Equations - Chapter 10 (One week)
- Three-Dimensional Space - Chapter 12 (Four weeks)
- Curves in 3D - Chapter 13 (Two weeks)
- Multivariate Functions (Derivatives) - Chapter 14 (Four weeks)
- Multivariate Functions (Integration) - Chapter 15 (Four weeks)

It is expected that you have obtained proficiency content of Math 30 and Math 31 and, in particular, with basic differentiation and integration techniques. We will not have time to review these topics! These fundamental concepts will factor significantly in our analysis of higher dimensional equations. A background in vector algebra along the lines of Math 129A would be beneficial but is in no way required. We will cover the required geometric notions in some detail when the arise.
Section 2: Parametric Equations

We are used to seeing relationships in either the functional form $y = f(x)$, which expresses that $y$ depends explicitly on $x$, or the relational form $f(x, y) = C$, which expresses that $y$ depends implicitly on $x$. The relational form is required when the graph does not pass the vertical line test, e.g. the relationship $x^2 + y^2 = 1$, which generates a circle of radius one.

For many applications, these representations are not ideal. For example, consider the following curve:

This curve cannot be represented in a functional form. It also does not permit a simple implicit formulation. Nevertheless, we can imagine the curve as representing real-world phenomena. It may correspond to the layout of a particular country road, or the flight path of an airplane. The question is how we represent these curves algebraically in a convenient way.

The answer is to represent the curve parametrically. In this setting, our $x$ and $y$ coordinates are controlled independently by some new independent variable (commonly denoted $t$). A conveniently interpretation of this process is to imagine moving along the curve from one end to the other while keeping track of how far we have traveled. We can then determine our coordinates in the $(x, y)$-plane as a function of how far we have traveled (the value of $t$):
Formally, we define the following.

**Definition 1**

A curve \( r(t) \) in the \((x,y)\)-plane is said to be in **parametric form** if it is represented as \( r(t) = (x(t), y(t)) \) where \( a \leq t \leq b \). The point \((x(a), y(a))\) is called the **initial point** and the point \((x(b), y(b))\) is called the **terminal point**.

Notice that, although the curve \( r(t) \) itself is not required to a function of \( x \) or \( y \), it is the case that both \( x \) and \( y \) are functions of \( t \). That is, for each value of the parameter \( t \), we must be able to determine a unique value for \( x \) and a unique value for \( y \).

**Example 1**

Plot the following curve \( r(t) \) in the \((x,y)\)-plane:

\[
r(t) = (t^2 + 1, t^2 - t), \quad -2 \leq t \leq 2.
\]

**Solution:** In a few lectures, we will discuss advanced analysis techniques (e.g. differentiation) which will give more concrete information about the shape of the curves. For now we simply build some intuition. We recognize that each value of \( t \) in the interval \([-2, 2] \) is associated to a point \((x(t), y(t))\) in the plane. To get a sense of the curve, we can draw the curves \( x(t) = t^2 + 1 \) and \( y(t) = t^2 - t \) separately and use this information to obtain a few points of interest and the general shape of the curve in the \((x,y)\)-plane. For obtain particular values, for instance, we can evaluate

\[
x(-2) = (-2)^2 + 1 = 4 + 1 = 5
\]
\[
y(-2) = (-2)^2 - (-2) = 6
\]

so that \( t = -2 \) is associated with the point \((5, 6)\). Evaluating for a few convenient values of \( t \) gives the following table:
<table>
<thead>
<tr>
<th>t</th>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>-1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>2</td>
</tr>
</tbody>
</table>

Even without information about how the curve winds through space, we can construct a reasonable representation of $\mathbf{r}(t)$ (right) by smoothly connecting the dots as follows, where the parabolas $x(t) = t^2 + 1$ and $y(t) = t^2 - t$ are also shown (left):

Example 2

Show that the following curve parameterizes the unit circle:

$$\mathbf{r}(t) = (\cos(t), \sin(t)), \quad 0 \leq t \leq 2\pi.$$  

Solution: The equation for the unit circle (circle of radius one) is given by $x^2 + y^2 = 1$. We simply need to plug the given expressions for $x$ and $y$ into this equation. We have

$$x^2 + y^2 = \cos^2(t) + \sin^2(t) = 1$$

by fundamental trigonometric identities, and we are done.
Example 3

Plot the following curve \( r(t) \) in the \( (x, y) \)-plane:

\[
r(t) = (\cos(t), -\sin^2(t) + 1), \quad 0 \leq t \leq \pi.
\]

Also determine an equation for this curve consisting of only \( x \) and \( y \) (i.e. eliminate \( t \)).

Solution: We compute a few points as before to obtain the following:

<table>
<thead>
<tr>
<th>( t )</th>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \frac{\pi}{4} )</td>
<td>( \frac{1}{\sqrt{2}} )</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td>( \frac{\pi}{2} )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \frac{3\pi}{4} )</td>
<td>( -\frac{1}{\sqrt{2}} )</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td>( \pi )</td>
<td>( -1 )</td>
<td>1</td>
</tr>
</tbody>
</table>

To get a sense for the shape of the curve, we play the functions \( x(t) = \cos(t) \) and \( -2 \sin(t) + 1 \). This yields the following:

The second part of the question asks us to express the relationship in terms of \( x \) and \( y \) only. We can accomplish this by eliminating \( t \) from either equation, then substituting it into the other equation. We have

\[
x = \cos(t) \implies t = \arccos(x).
\]

We can substitute this into the relationship for \( y \) to get

\[
y = -\sin^2(\arccos(x)) + 1 = -\left(\sqrt{1 - x^2}\right)^2 + 1 = x^2
\]
where we have evaluated \( \sin(\arccos(x)) = \sqrt{1-x^2} \) by appealing to trigonometric triangles. This result is interesting: it tells us that the parametric expressions for \( r(t) \), which contained trigonometric functions, simplify to the parabola \( y = x^2 \). This perhaps should not have be surprising consider the plot we have obtained.

Notice that we can also verify that this equation works by plugging the original expressions for \( x \) and \( y \) into the derived expression. We have

\[
y = -\sin^2(t) + 1 = -(1 - \cos^2(t)) + 1 = \cos^2(t) = x^2.
\]

**Note:** We should not get too comfortable with parametric equations corresponding to relationships which are easily representable in functional form. Nevertheless, Example 3 demonstrates the methodology for returning from parametric form to implicit form. We must eliminate \( t \) from one of the expression and substitute this expression into the remaining equation.

**Suggested Problems**

1. Plot the following curves \( r(t) \) in the \((x, y)\)-plane:

   (a) \[
   \begin{cases}
   r(t) = (t + 1, 3 - 2t) \\
   -1 \leq t \leq 3
   \end{cases}
   \]

   (b) \[
   \begin{cases}
   r(t) = (t^2 - t, -t^2 - 2t) \\
   -2 \leq t \leq 2
   \end{cases}
   \]

   (c) \[
   \begin{cases}
   r(t) = (\cos(t), \sin(2t)) \\
   0 \leq t \leq \pi
   \end{cases}
   \]

   (d) \[
   \begin{cases}
   r(t) = (2t^3 - 1, t^2 - 1) \\
   -1 \leq t \leq 1
   \end{cases}
   \]

2. Represent the following as implicit relationships of \( x \) and \( y \):

   (a) \( r(t) = (e^t - t, t) \)

   (b) \( r(t) = (\ln(t), t \ln(t)) \)

   (c) \( r(t) = (\sin(2t), 2 \cos^2(t) - 1) \)

   (d) \[
   r(t) = \left( \frac{1 - t^2}{1 + t^2}, \frac{-2t}{1 + t^2} \right)
   \]

   [\text{Hint: Consider squaring the expressions for } x \text{ and } y!]\n
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