Section 1: Polar Graphing

Our approach to graphing polar functions will be very similar to what we used when graphing parametric equations. We start by plotting the function \( r(\theta) \) and determining a few crucial points \( \theta \) and \( r \) in a table.

Our interpretation when graphing, however, will be different than it was for parametrization. We will think of \( \theta \) as parametrizing an array extending from \((0, 0)\) to infinity in some direction, where \( \theta \) is the angle between the array and the positive \( x \)-axis. As the array pivots upon the axis (i.e. as \( \theta \) increases), we assign points on that array at the appropriate distance \( r \) given by the function \( r(\theta) \).

**Example 1**

Graph \( r = \sin(2\theta) \) using polar coordinates.

**Solution:** When we transformed this equation into Cartesian coordinates, it did not fit the form of any function we recognize—polar coordinates is the best we can do. We start by plotting \( r = \sin(2\theta) \) and making a table of critical values over the interval \( 0 \leq \theta \leq 2\pi \):

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( r )</th>
<th>( \theta )</th>
<th>( r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>( \pi )</td>
<td>0</td>
</tr>
<tr>
<td>( \pi/4 )</td>
<td>1</td>
<td>( 5\pi/4 )</td>
<td>1</td>
</tr>
<tr>
<td>( \pi/2 )</td>
<td>0</td>
<td>( 3\pi/2 )</td>
<td>0</td>
</tr>
<tr>
<td>( 3\pi/4 )</td>
<td>-1</td>
<td>( 7\pi/4 )</td>
<td>-1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( 2\pi )</td>
<td>0</td>
</tr>
</tbody>
</table>

Consider the region \( 0 \leq \theta \leq \pi/2 \) (i.e. the first quadrant). The radius \( r \) goes from \( r = 0 \) at \( \theta = 0 \) (the positive \( x \)-axis) to \( r = 1 \) at \( \theta = \pi/4 \) and then...
back to $r = 0$ at $\theta = \frac{\pi}{2}$. This sweeps out what looks like the “petal” of a flower. As we sweep out the next quadrant of values $\frac{\pi}{2} \leq \theta \leq \pi$, we see that the same pattern repeats; however, the radius is negative, so that it sweeps out a “petal” in the fourth quadrant. Repeating the procedure for $0 \leq \theta \leq 2\pi$ we see that all four quadrants are filled in to give what appears to be a four-petalled flower:

Example 2

Graph $r = \cos(\theta) - 2\sin(\theta)$ using polar coordinates.

Solution: Our table of important points is as follows:

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0$</td>
<td>$1$</td>
</tr>
<tr>
<td>$\frac{\pi}{2}$</td>
<td>$-2$</td>
</tr>
<tr>
<td>$\pi$</td>
<td>$-1$</td>
</tr>
</tbody>
</table>

We could go further; however, we notice that $(1, 0)$ and $(-1, \pi)$ in polar coordinates correspond to the same point in Cartesian coordinates, namely, $(1, 0)$. Parametrizing $\theta$ through $[\pi, 2\pi]$ would repeat the same points we have already attained. We consider the point where $r = 0$ as important since it defines a tipping point where $r$ goes from being positive to being negative.

To get the shape of the function, we want to graph $r = \cos(\theta) - 2\sin(\theta)$; however, there is no clear way to accomplish this. We will need
to represent this expression in a different form. In fact, we have that
\[ r = \cos(\theta) - 2 \sin(\theta) = \sqrt{5} \cos(\theta - \arctan(-2)). \]

This can be seen by noting that
\[ A \cos(\theta - \phi) = A \cos(\phi) \cos(\theta) + A \sin(\phi) \sin(\theta). \]

In order to equate this with the original expression for \( r \), we require
\[ A \cos(\phi) = 1 \]
\[ A \sin(\phi) = -2. \]

We have already discussed how to solve a system of this form when we discussed the transformation into polar coordinates (we have \( r = A \), \( \theta = \phi \), \( x = 1 \), and \( y = -2 \)). Solving for \( A \) and \( \phi \) and checking quadrants yields the required form. [Note that \( \arctan(-2) \approx -1.107 \) so that \( \sqrt{5} \cos(\theta - \arctan(-2)) \) is a cosine function with magnitude \( \sqrt{5} \) shifted to the left 1.107 units.]

We can now obtain an idea of what the function looks like by plotting \( r = \sqrt{5} \cos(\theta - \arctan(-2)) \):

Notice that the value of \( r \) shifts from positive to negative between \( \theta = 0 \) and \( \theta = \frac{\pi}{2} \). We could obtain the exact value of \( \theta \) where this tipping point occurs, but it does not matter for the qualitative picture. Putting everything together, we have the circle centred at \( (1/2, -1) \) of radius \( \sqrt{5}/4 \), as we expected from last lecture.
Example 3

Graph the following polar function:

\[ r = \theta, \quad \theta \geq 0. \]

**Solution:** We notice first of all that, since \( \theta \) does not appear on the inside of trigonometric function, we are going to have some difficulty changing this into Cartesian coordinates. We have to approach this in polar coordinates, but we are at a loss for which critical values of \( \theta \) to put in our chart.

Fortunately, things are actually much easier for this example than in the previous ones. As \( \theta \) grows, so does the radius. So our curve starts out at the original and slowly rotates counterclockwise, growing at an even rate. This is clearly a *spiral*. 

![Graph of polar function](image)
Section 2: Polar Derivatives

When we considered single-variable functions, a helpful aid for graphing and analysis came from evaluating derivatives. Critical points corresponded to potential maxima and minima, and inflection points corresponded to points where the concavity changed. We are still interested in these topics with parametric and polar equations, but information about slopes and concavity in the \((x, y)\)-plane are not immediately apparent from parametric equations.

To update our techniques to polar equations, we take the form \(r = f(\theta)\) and note that we can represent points as

\[
(x, y) = (r \cos(\theta), r \sin(\theta)) = (f(\theta) \cos(\theta), f(\theta) \sin(\theta)).
\]

We can now apply our differentiation formulas to obtain the following.

**Polar Derivatives:***

\[
\frac{dy}{dx} = \frac{dy}{d\theta} \frac{d\theta}{dx} = \frac{f'(\theta) \sin(\theta) + f(\theta) \cos(\theta)}{f'(\theta) \cos(\theta) - f(\theta) \sin(\theta)}.
\]

**Example 4**

Use polar coordinates to find all critical points of a circle of radius \(\rho\).

**Solution:** In polar coordinates, the equation for a circle of radius \(\rho\) is \(r = f(\theta) = \rho\). It seems ridiculous to take a derivative of this expression, since it is a constant, but our formula for the derivative \(y'(x)\) requires more information. In fact, we have

\[
\frac{dy}{dx} = \frac{f'(\theta) \sin(\theta) + f(\theta) \cos(\theta)}{f'(\theta) \cos(\theta) - f(\theta) \sin(\theta)} = \frac{-\rho \cos(\theta)}{\rho \sin(\theta)} = \cot(\theta).
\]

We know that \(y'(x) = \cot(\theta) = 0\) corresponds to \(\theta = \pi/2\) and \(\theta = 3\pi/2\).
It follows that we have the points

\[
\theta = \frac{\pi}{2} \implies x\left(\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}\right) \cos\left(\frac{\pi}{2}\right) = 0
\]

\[
y\left(\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}\right) \sin\left(\frac{\pi}{2}\right) = \rho
\]

\[
\theta = \frac{3\pi}{2} \implies x\left(\frac{3\pi}{2}\right) = f\left( \frac{3\pi}{2}\right) \cos\left(\frac{3\pi}{2}\right) = 0
\]

\[
y\left(\frac{3\pi}{2}\right) = f\left(\frac{3\pi}{2}\right) \sin\left(\frac{3\pi}{2}\right) = -\rho.
\]

It follows the points we are interested in are \((0, \rho)\) and \((0, -\rho)\), which is consistent with our intuition that the arc of a circle flattens out at the top and the bottom.

**Example 5**

Find the Cartesian coordinates all critical points of

\[ r = \sin(2\theta) \]

**Solution:** We are looking for points where \( y'(x) = 0 \). We apply our formula to get

\[
\frac{dy}{dx} = \frac{2 \cos(2\theta) \sin(\theta) + \sin(2\theta) \cos(\theta)}{2 \cos(2\theta) \cos(\theta) - \sin(2\theta) \sin(\theta)}.
\]

In order to have a horizontal tangent line, we only need to set the numerator to zero, so that we have

\[
0 = 2 \cos(2\theta) \sin(\theta) + \sin(2\theta) \cos(\theta)
\]

\[
= 2(1 - 2 \sin^2(\theta)) \sin(\theta) + 2 \sin(\theta) \cos^2(\theta)
\]

\[
= 2 \sin(\theta) - 4 \sin^3(\theta) + 2 \sin(\theta)(1 - \sin^2(\theta))
\]

\[
= 4 \sin(\theta) - 6 \sin^3(\theta)
\]

\[
= 2 \sin(\theta)(2 - 3 \sin^2(\theta)).
\]
This is satisfied if either

\[
\sin(\theta) = 0, \quad \text{or} \quad \sin(\theta) = \pm \sqrt{\frac{2}{3}}.
\]

The first is satisfied if

\[\theta = n\pi, \quad n \text{ an integer}\]

while the second can be satisfied if either

\[\theta = \arcsin(\sqrt{\frac{2}{3}}) + n\pi \quad \text{or} \quad \theta = \arcsin(-\sqrt{\frac{2}{3}}) + n\pi, \quad n \in \mathbb{Z}.\]

Notice that \(\theta = n\pi\) gives \(r = 0\) for all \(n\) and therefore corresponds to the Cartesian point \((0,0)\). We will therefore count it only once and set \(\theta_1 = 0\). Since \(\theta\) repeats after \(2\pi\), we will reduce the remaining list of angles to the following:

\[
\theta_2 = \arcsin(\sqrt{\frac{2}{3}}), \quad \theta_3 = \arcsin(\sqrt{\frac{2}{3}}) + \pi,
\]

\[
\theta_4 = \arcsin(-\sqrt{\frac{2}{3}}), \quad \theta_5 = \arcsin(-\sqrt{\frac{2}{3}}) + \pi.
\]

Since \(\theta\) parametrizes the equation, we can use the formulas to get

\[
x = r \cos(\theta) = f(\theta) \cos(\theta) = 2 \sin(\theta) \cos^2(\theta)
\]

\[
y = r \sin(\theta) = f(\theta) \sin(\theta) = 2 \sin^2(\theta) \cos(\theta).
\]

We can use triangles to solve for the required values of \(\sin(\theta)\) and \(\cos(\theta)\). We have

\[
\sin\left(\arcsin\left(\sqrt{\frac{2}{3}}\right)\right) = \sqrt{\frac{2}{3}}
\]

\[
\cos\left(\arcsin\left(\sqrt{\frac{2}{3}}\right)\right) = \frac{1}{\sqrt{3}}
\]

\[
\sin\left(\arcsin\left(-\sqrt{\frac{2}{3}}\right)\right) = -\sqrt{\frac{2}{3}}
\]

\[
\cos\left(\arcsin\left(-\sqrt{\frac{2}{3}}\right)\right) = \frac{1}{\sqrt{3}}.
\]
We can now solve for the Cartesian points \((x, y)\) to obtain:

\[
\begin{align*}
\theta_1 &= 0 : \quad (x_1, y_1) = (0, 0) \\
\theta_2 &= \arcsin(\sqrt{2}/3) : \quad (x_2, y_2) = \left(\frac{2\sqrt{2}}{3\sqrt{3}}, \frac{4}{3\sqrt{3}}\right) \\
\theta_3 &= \arcsin(\sqrt{2}/3) + \pi : \quad (x_3, y_3) = \left(-\frac{2\sqrt{2}}{3\sqrt{3}}, -\frac{4}{3\sqrt{3}}\right) \\
\theta_4 &= \arcsin(-\sqrt{2}/3) : \quad (x_4, y_4) = \left(-\frac{2\sqrt{2}}{3\sqrt{3}}, \frac{4}{3\sqrt{3}}\right) \\
\theta_5 &= \arcsin(-\sqrt{2}/3) + \pi : \quad (x_5, y_5) = \left(\frac{2\sqrt{2}}{3\sqrt{3}}, -\frac{4}{3\sqrt{3}}\right)
\end{align*}
\]

This matches our graphical intuition since each petal of \(r = 2\sin(\theta)\) contains a single point where the derivative is zero. We have also shown that the curves pass through \((0, 0)\) horizontally, which it does every time the function swings from left to right or vice-versa:
1. Plot the following polar relationships for $0 \leq \theta \leq 2\pi$ and determine all critical points:

(a) $r = \tan(\theta)$
(b) $r = 3\cos(\theta) + 4\sin(\theta)$
(c) $r = -\cos(\theta) + \sin(\theta)$
(d) $r = \frac{1}{1 + \sin(\theta)}$
(e) $r = \cos(3\theta)$ [Hint: Consider how the function repeats periodically!]
(f) $r = 1 + \sin(2\theta)$
(g) $r = \theta \sin(\theta)$